Final round
Dutch Mathematical Olympiad

Friday 13 September 2024

Solutions

1. Suppose problem A is on tables A1 and A2, problem B is on tables B1 and B2, and problem C is on tables C1 and C2. Consider the students sitting at table A1 in the first round. In the second round, they have to go to a table that does not have problem A on it, i.e. table B1, B2, C1 or C2. Moreover, these students all have to go to a different table. So there cannot be more than four students at table A1 in the first round. This applies to all six tables, so no more than $4 \cdot 6 = 24$ students can participate.

To show that the maximum of 24 students can also be met, we make a schedule for the three rounds.

	A1		A2		B1		B2		C1		C2	
Round 1	1	2	5	6	9	10	13	14	17	18	21	22
	3	4	7	8	11	12	15	16	19	20	23	24
Round 2	9	13	10	14	1	5	2	6	3	7	4	8
	17	21	18	22	19	23	20	24	11	15	12	16
Round 3	11	16	12	15	3	8	4	7	1	6	2	5
	19	24	20	23	17	22	18	21	9	14	10	13

One method of making such a schedule is to first distribute students 1 to 8, who do task A in the first round, to the other four tables in the second and third rounds. Then you can use exactly the same distribution, but with the tasks swapped, for students 9 to 16 who start with task B. Finally, you do the same again with students 17 to 24 who start with task C, who then fit exactly into the remaining spots.

2. Version for klas 4 and below

(a) We are going to prove that for every n, Mila can return to her starting square after an $(n+1)$-jump, $(n+2)$-jump, $(n+3)$-jump and $(n+4)$-jump. Suppose Mila starts on the square with coordinates $(0,0)$. Then Mila can return by first jumping to $(1, n+1)$, then to $(n+3, n+2)$, then to $(n+4, -1)$ and finally back to $(0,0)$. The desired statement now follows from taking $n = 100$.

(b) Colour the squares on the board alternately white and black, like on a chess board. If n is odd, then an n-jump always goes to a square of the same colour as the starting square, and if n is even precisely to a square of the other colour. Suppose Mila starts on a white square. After the jumps, Mila ends up successively on squares with colours white, black, black, white, white, black, black, white, and so on.

More precisely, if Mila makes a total of m jumps and starts on a white square, Mila ends on a white square if m is of the form $4k$ or $4k+1$, where k is an integer, and on a black square if m is of the form $4k+2$ or $4k+3$.

So the only possibilities for Mila to end up on the initial square are for m of the form $4k$ or $4k+1$. In the first case, we see immediately from part (a) that this is indeed possible. For $m = 1$ it is not possible to end on the square where Mila started, but for $m = 5$ it can be done by jumping as follows:

$$(0,0) - (1,1) - (2,3) - (5,4) - (1,5) - (0,0).$$

Then by pasting the path from part (a) behind this for $n = 5, 9, 13, \ldots$, we see that it is thus possible for Mila to end on the square she started, for m of the form $4k+1$, as long as $m \geqslant 5$.

Together we find that Mila can return to her starting square after m jumps for every m of the form $4k$ or $4k+1$, for $m \geqslant 4$ (and thus $k \geqslant 1$).

2. Version for klas 5 & klas 6

In the solution for klas 4 and below, part (a) proves a simpler case. Part of this is proving that for every n, Mila can return to her starting square after an $(n+1)$-jump, $(n+2)$-jump, $(n+3)$-jump and $(n+4)$-jump. This result is used in (b) to prove that Mila can return to her starting square after m jumps for every m of the form $4k$ or $4k+1$, for $m \geqslant 4$ (and thus $k \geqslant 1$).

3. The smallest integer greater than 1 is 2, so we see that

$$\frac{b^3}{a^4} \geqslant 2 \qquad \text{and} \qquad \frac{a^3}{b^2} \geqslant 2.$$

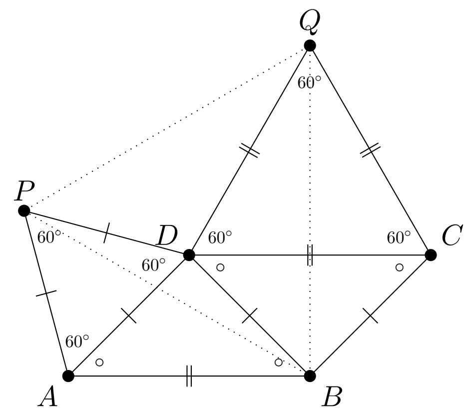
It follows that

$$a = \left(\frac{b^3}{a^4}\right)^2 \cdot \left(\frac{a^3}{b^2}\right)^3 \geqslant 2^5 \qquad \text{and} \qquad b = \left(\frac{b^3}{a^4}\right)^3 \cdot \left(\frac{a^3}{b^2}\right)^4 \geqslant 2^7.$$

So we see that $a + b \geqslant 2^5 + 2^7 = 160$. On the other hand, we see that $(a, b) = (2^5, 2^7)$ is a solution. So 160 is the smallest possible value for $a + b$.

4. Version for klas 5 & klas 4 and below

(a) There are a lot of line segments of the same length. For example, $|AD| = |DP| = |PA|$ because triangle $\triangle ADP$ is equilateral. It is given that $|AD| = |BD|$, and finally $|AD| = |BC|$ because $ABCD$ is a parallelogram. Similarly, $|CD| = |DQ| = |QC| = |AB|$. The opposite angles in the parallelogram are the same size, so $\angle DAB = \angle BCD$. Since $\triangle ABD$ and $\triangle DBC$ are isosceles triangles, these angles are also equal to $\angle ABD$ and $\angle CDB$. Finally, the angles of the equilateral triangles $\triangle ADP$ and $\triangle CDQ$ are all $60°$. All this information is summarised in the figure below.



Since $|PA| = |BC|$ and $\angle PAB = \angle BCQ$ and $|AB| = |CQ|$, $\triangle PAB$ and $\triangle BCQ$ are congruent triangles. It follows that $|PB| = |BQ|$.

(b) Since the angles at D add up to $360°$ and the non-opposing angles of the parallelogram add up to $180°$, we find that

$$\begin{aligned}\angle PDQ &= 360° - 2 \cdot 60° - \angle CDA \\ &= 240° - (180° - \angle DAB) \\ &= 60° + \angle DAB \\ &= \angle PAB.\end{aligned}$$

Since also $|PA| = |PD|$ and $|AB| = |DQ|$, we find that also triangle $\triangle PDQ$ is congruent with $\triangle PAB$ and $\triangle BCQ$. Thus, triangle $\triangle PBQ$ is equilateral, and has angles of $60°$.

For the last step, we use that line segment QB is the perpendicular bisector of DC, because B and Q are both equally far from C and from D. Since triangle $\triangle CDQ$ is equilateral, QB is also the bisector of $\angle CQD$. It follows that $\angle DQB = 30°$, and so $\angle PQD = 60° - 30° = 30°$.

4. Version for klas 6

In the solution for klas 5 & klas 4 and below the problem is solved in two parts. Part (a) proves the intermediate step $|PB| = |BQ|$; part (b) solves the problem and proves that $\angle PQD = 30°$.

5. Version for klas 4 and below

(a) Suppose that n is a self-squared ℓ-code. Then the last ℓ digits of $n^2 - n = n(n-1)$ are all zeros. This means that $n(n-1)$ is divisible by 10^ℓ. Vice versa, n is self-squared if $n^2 - n$ is divisible by 10^ℓ and thus ends in ℓ zeros.

(b) The number $n(n-1)$ is divisible by 10 and so, in particular, it is divisible by 5. This means that one of the numbers n and $n-1$ must be divisible by 5. In other words, the last digit of either n or $n-1$ must be a 0 or 5. So the only possible final digits for n are 0, 1, 5 and 6.

If the last digit of n is 0, then $n-1$ is not divisible by 2 or 5. Since $n(n-1)$ is divisible by $10^\ell = 2^\ell \cdot 5^\ell$ and there can be no factors 2 or 5 in $n-1$, n must be divisible by 10^ℓ. But that means that the last ℓ digits of n, which are all the digits of n, must all be zeros. Then we get $n = 0$, which is smaller than 2.

If the last digit of n is a 1, then n is not divisible by 2 or 5. Since $n(n-1)$ is divisible by $10^\ell = 2^\ell \cdot 5^\ell$ and there can be no factors 2 or 5 in n, $n-1$ must be divisible by 10^ℓ. But that means that the last ℓ digits of $n-1$, which are all the digits of $n-1$, must all be zeros. Then we get $n = 1$, which is smaller than 2.

So all self-quadratic codes $n \geqslant 2$ end in a 5 or 6.

(c) Suppose that $m = c \cdot 10^\ell + n$ is an $(\ell+1)$-code obtained by putting c in front of n. Because of (a), m is self-squared if and only if

$$m(m-1) = (c \cdot 10^\ell + n)(c \cdot 10^\ell + n - 1) = c^2 \cdot 10^{2\ell} + c \cdot 10^\ell(2n-1) + n(n-1)$$

is divisible by $10^{\ell+1}$. As $\ell \geqslant 1$, the term $c^2 \cdot 10^{2\ell}$ is also divisible by $10^{\ell+1}$. The number $n(n-1)$ is divisible by 10^ℓ, because of part (a), and can therefore be written as $d \cdot 10^\ell$. It then follows that

$$c \cdot 10^\ell(2n-1) + d \cdot 10^\ell = 10^\ell \cdot (c(2n-1) + d)$$

must be divisble by $10^{\ell+1}$, or in other words that $c(2n-1) + d$ must be divisible by 10.

Because of part (b), n must end in a 5 or 6. If n ends in a 5, then $2n-1$ is a multiple of ten minus one, i.e. it is of the shape $10k-1$, and we see that $c(10k-1)+d$ must be divisible by 10 and so $-c+d$ must be divisible by 10. On the other hand, if n ends in a 6, then $2n-1$ is a multiple of ten plus one, say $10k+1$, and we see that $c(10k+1)+d$ must be divisible by 10 and so $c+d$ must be divisible by 10.

In both cases, there is a unique possibility for c from the numbers $0, 1, 2, \ldots, 9$ for which $-c+d$ (or $c+d$) is divisible by 10, since, of the ten consecutive numbers $-0+d, -1+d, \ldots, -9+d$ (or $0+d, 1+d, \ldots, 9+d$), exactly one is divisible by 10. Therefore, there is a unique digit c that we can put before the ℓ-code n such that m is a self-squared $(\ell+1)$-code.

5. Version for klas 5 & klas 6

The solutions for parts (a), (b) and (c) can be found at the version for klas 4 and below.

(d) Suppose that $n \geqslant 2$ is a self-squared ℓ-code. Then we will show that $k = 10^\ell + 1 - n$ is also a self-squared ℓ-code. Because $n \geqslant 2$, we have $k \leqslant 10^\ell - 1$ and k can thus really be written using ℓ digits. From part (a) it follows that k is self-squared if and only if

$$k(k-1) = (10^\ell + 1 - n)(10^\ell - n) = 10^{2\ell} + 10^\ell(1 - 2n) + (n-1)n$$

is divisible by 10^ℓ. The first two terms are always divisible by 10^ℓ and $(n-1)n$ is divisible by 10^ℓ because n is self-squared. So it follows that k is self-squared.

Since $n \leqslant 10^\ell - 1$, we have $k \geqslant 2$. Furthermore, $10^\ell + 1$ is odd, and so one out of n and k is even and the other number is odd. In particular, n and k are thus not equal to each other. Since there are exactly two self-squared ℓ-codes $\geqslant 2$, the other self-squared ℓ-code m must be equal to k. It then holds that $m + n = k + n = 10^\ell + 1$.