Final round
Dutch Mathematical Olympiad

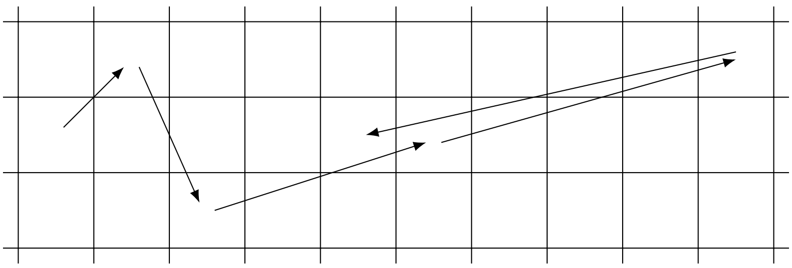
Version klas 5

Friday 13 September 2024
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Formula sheets, calculators, and other electronic devices are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. A teacher wants to spend a morning practising Olympiad problems with her class in teams. To this end, she has set up an arrangement with six large tables. At each table several students can work together on a problem. Each table has a problem; there are three different problems, each on two tables. There are three rounds in which each student sits at one of the tables. The teacher wants to schedule the students in such a way that after three rounds, each student has worked on all the problems once. In addition, she wants each pair of students to sit at the same table at most once.
What is the maximum number of students for which such a schedule is possible?

2. Mila stands on an infinitely large board divided into squares and starts moving. An n-jump is a movement in which Mila moves one square left, right, up or down and then n squares in a direction perpendicular to that. Below is an example where Mila starts in the middle box on the left and first does a 1-jump, followed by a 2-jump and then a 3-jump, 4-jump and 5-jump.



Suppose Mila first does a 1-jump, then a 2-jump, then a 3-jump, a 4-jump, and so on. Finally, she does a m-jump. For which positive integers m can Mila choose her m jumps such that she can get back to her starting square?

3. Given are two positive integers a and b with the property that

$$\frac{b^3}{a^4} \quad \text{and} \quad \frac{a^3}{b^2}$$

are both integers greater than 1.
What is the smallest possible value for the sum $a + b$?

PLEASE CONTINUE ON THE OTHER SIDE

4. Let $ABCD$ be a parallelogram with the property that $|AD| = |BD|$. Now let P and Q be points such that $\triangle ADP$ and $\triangle CDQ$ are equilateral and do not overlap with the parallelogram.

(a) Prove that $|PB| = |BQ|$.

(b) Prove that $\angle PQD = 30°$.

5. An ℓ-code is an integer $n \geqslant 0$ of at most ℓ digits, if necessary supplemented by leading zeros, so that it consists of ℓ digits in total. Thus, you can make a 4-code out of 310 by writing it as 0310. An ℓ-code is called *self-squared* if the last ℓ digits of the square of that code form exactly the original ℓ-code. Thus, the 1-codes 5 and 6 are self-squared, but the 3-code 006 is not self-squared, because $6^2 = 36 = 036$ does not end in 006. The 2-code 76 is self-squared, because $76^2 = 5776$ ends in 76.

(a) Prove that an ℓ-code n is self-squared if and only if $n(n-1)$ is divisible by 10^ℓ.

(b) Prove that any self-squared ℓ-code $n \geqslant 2$ ends in the digit 5 or the digit 6.

(c) Prove that any self-squared ℓ-code $n \geqslant 2$ is extendable to a self-squared $(\ell + 1)$-code in exactly one way by placing a digit in front of it.

(d) It follows from (b) and (c) that for every ℓ, there exist exactly two self-squared ℓ-codes $m, n \geqslant 2$. What is their sum $m + n$ (in terms of ℓ)?