Final round
Dutch Mathematical Olympiad

Friday 15 September 2023

Solutions

1. Version for klas 4 and below

If $[a \cdot b] = 2$, then $a \cdot b$ equals 2, 20, 200, or 2000. Bigger is not possible, because $a, b < 100$ so $a \cdot b < 10000$. We will look at the prime decomposition of these four possibilities to see what options there are for a and b.

Since a and b are both greater than 1, $a \cdot b = 2$ is not possible. For $a \cdot b = 20 = 2 \cdot 2 \cdot 5$, we find the possibility $(a, b) = (4, 5)$. Since a and b are nillless, $(a, b) = (2, 10)$ and $(a, b) = (1, 20)$ are impossible. Then we look at $a \cdot b = 200 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$. Due to their nilllessness both a and b cannot have a factor 2 and a factor 5 at the same time. Numbers that do have those, are divisible by 10 and hence end in a 0. So the only solution with $a \cdot b = 200$ is $(a, b) = (8, 25)$. Finally, we consider $a \cdot b = 2000 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$. The only nillless factorisation is $(a, b) = (16, 125)$, but that contradicts $b < 100$.

So the only two solutions are $(a, b) = (4, 5)$ and $(a, b) = (8, 25)$.

1. Version for klas 5 & klas 6

If $[a \cdot b] - 1$ is a nillless number, then it follows from $[[a \cdot b] - 1] = 1$ that $[a \cdot b] = 2$. This case was covered above in the solution for klas 4 and below. Now assume that $[a \cdot b] - 1$ is not nillless. The difference is unequal to 1, so $[a \cdot b] - 1$ is equal to 10, 100, 1000, et cetera, and hence $[a \cdot b]$ is equal to 11, 101, 1001, et cetera. But since $[a \cdot b]$ does not contain zeros, we only have the option $[a \cdot b] = 11$. Hence, $a \cdot b$ is a number consisting of two ones and some zeros. Since $a, b < 100$ we have $a \cdot b < 10000$, so $a \cdot b$ consists of at most four digits. We look at all the possibilities and find a nillless factorisation.

- $a \cdot b = 11$ and $a \cdot b = 101$ are not possible, because those are prime numbers and $a > 1$. In the following we disregard factorisations with $a = 1$.
- $a \cdot b = 110 = 2 \cdot 5 \cdot 11$ gives solutions $(a, b) = (2, 55)$ and $(a, b) = (5, 22)$. The option $(a, b) = (10, 11)$ is not possible because 10 is not nillless.
- $a \cdot b = 1001 = 7 \cdot 11 \cdot 13$ gives solutions $(a, b) = (11, 91)$ and $(a, b) = (13, 77)$. The option $(a, b) = (7, 143)$ is not possible because it does not have $b < 100$.
- $a \cdot b = 1010 = 2 \cdot 5 \cdot 101$ also has three factorisations: $2 \cdot 505$, $5 \cdot 202$ and $10 \cdot 101$. None of them is nillless.
- $a \cdot b = 1100 = 2 \cdot 2 \cdot 5 \cdot 5 \cdot 11$ gives only the solution $(a, b) = (25, 44)$. We already saw that a and b both do not have a factor 2 and a factor 5, so the only other option is $(a, b) = (4, 275)$ and this is a contradiction with $b < 100$.

In total, we find seven solutions: $(4, 5)$, $(8, 25)$, $(2, 55)$, $(5, 22)$, $(11, 91)$, $(13, 77)$, and $(25, 44)$.

2. A possible way to provide each vase with a note is to put in vase 1 a note with 1, in vase 2 a note with 2, in vase 3 a note with 3, ..., and in vase 2023 a note with 2023. We will use induction to show that this is the only distribution. Note that for a valid distribution it does not matter in which order we fill the vases.

Take a look at vase 1. Suppose we put a note in it with a. Then in vase a we put a note with b, such that $(a + b)/2 = 1$ or $a + b = 2$. Since a and b are positive integers, it must hold that $a = b = 1$. So the two vases here were the same and vase 1 contains a note with 1.

For the induction step, we assume that the first $n - 1$ vases each contain a note with the number of the vase. We want to show that in vase n we have to put a note with n. Suppose in vase n we put a note with a. If $a < n$, then vase a also contains a note with a because of the induction hypothesis. It must then hold that $(a + a)/2 = n$, but this contradicts $a < n$. We conclude that $a \geqslant n$.

If $a > n$, we do not know yet which number has to go on the note in vase a. Call this number b. Then it must hold that $(a + b)/2 = n$, and so $b < n$. But then we find a contradiction if we were to consider vase a first: in it we find a note with b, and in vase b we then find, because of the induction hypothesis, another note with b. However, $(b + b)/2 \neq a$, because $a > n$ and $b < n$.

We conclude that $a = n$ must hold: in vase n we also put a note with n on it. Induction now gives that in each vase we put a note with the number of the vase on it. So this is the only possible distribution of the notes.

3. Version for klas 4 and below

(a) Suppose Felix starts with an odd number n. Then the next number he writes down is $n^2 + 3$. We must show that $n^2 + 3$ is even, and $\frac{1}{2}(n^2 + 3)$ is even as well. In other words, $n^2 + 3$ must be divisible by 4. Since n is odd, we can write $n = 2k + 1$ for a non-negative integer k. Then it holds that

$$n^2 + 3 = (2k + 1)^2 + 3 = 4k^2 + 4k + 4 = 4(k^2 + k + 1).$$

and this is indeed divisible by 4.

(b) Suppose Felix starts with an odd number. Then the next number he writes down is divisible by 4, as we saw in part (a). So the next two numbers Felix gets by dividing by 2 each time. The result, $k^2 + k + 1$, is an odd number, independent of k being even or odd, as one of the two factors of $k^2 + k = k(k + 1)$ is even. Then the same three steps follow again: squaring plus 3, dividing by 2, and dividing by 2 again, and these three steps keep repeating. We are going to show that for odd $n \geqslant 5$, the odd number after three steps is greater than n. In that case, the process is going to produce larger and larger numbers and eventually Felix will write down a number of more than four digits. The inequality we want to check is $\frac{1}{4}(n^2 + 3) > n$, or, equivalently, $n^2 - 4n + 3 > 0$, or, equivalently, $(n - 2)^2 > 1$. This is indeed true for $n \geqslant 5$. Only for the odd starting numbers 1 and 3 do the numbers remain small: this gives the repetitions $1 - 4 - 2 - 1$ and $3 - 12 - 6 - 3$.

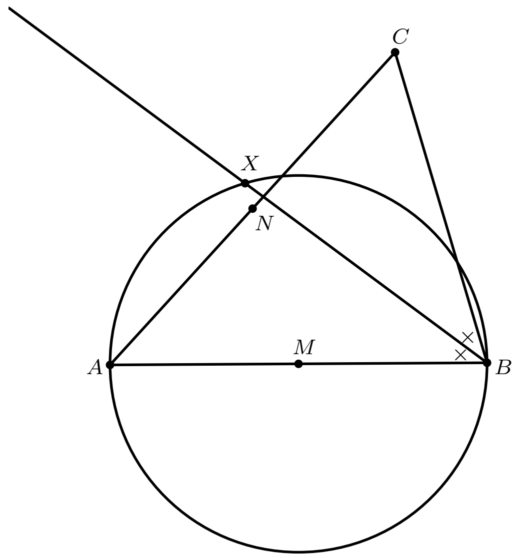
Suppose Felix starts with an even number. Then he starts dividing by 2 until he encounters an odd number. If that odd number is at least 5, then the numbers after that keep getting bigger. Only if repetitive dividing by 2 ends up with 1 or 3, the numbers after that always remain less than four digits. Those are numbers of the form 2^i or $3 \cdot 2^i$. Of the first form, 11 are smaller than 2023 ($2^0 = 1$ to $2^{10} = 1024$) and of the second form, 10 are smaller than 2023 ($3 \cdot 2^0 = 3$ to $3 \cdot 2^9 = 1536$). So in total there are 21 starting numbers where Felix will never write a number of more than five digits on the board.

3. Version for klas 5 & klas 6

In the solution for klas 4 and below, the question is solved in two parts. In part (a), we show that if Felix starts with an odd number, the next two numbers he writes down will be even. In part (b) we show that the number after that will be odd, from which we deduce that there are 21 starting numbers below 2023 where Felix will never write a number of more than four digits on the board.

4. Version for klas 5 & klas 4 and below

We will show that both MX and MN are parallel to BC. Since the two lines pass through the same point M, it follows that they are the same line and so M, N, and X lie on the same line.



To show that MX is parallel to BC, we note that BMX is an isosceles triangle with apex M. After all, MX and MB are the radius of the circle. It follows that $\angle MXB = \angle MBX$. Since also $\angle MBX = \angle XBC$, we have that $\angle MXB$ and $\angle XBC$ are alternate interior angles, and so MX and BC are parallel.

To show that MN is parallel to BC, we note that triangle ABC and AMN are similar, since $\frac{|AN|}{|AC|} = \frac{1}{2} = \frac{|AM|}{|AB|}$ and $\angle BAC = \angle MAN$. It follows that $\angle AMN = \angle ABC$ and because these are corresponding angles MN and BC are parallel.

4. Version for klas 6

Let M be the midpoint of AB. Now we can prove that the lines MX and MN are both parallel to BC; see the solutions for the version for klas 5 & klas 4 and below. It follows that MX and MN are in fact the same line, and that line is also the line XN. Hence, XN is parallel to BC.

5. Consider a sequence of cards for which exactly nine students get a turn. Now replace all cards k by $11 - k$. We will prove that this gives a sequence for which two students get a turn. We can repeat this process and that gives back the original sequence. Next, consider a sequence for which exactly two students get a turn and again replace all cards k by $11 - k$. We will prove that this gives a sequence for which nine students get a turn. Altogether, it will follow that $A = B$.

Suppose nine students get a turn. Then eight students each picked one card, and one student picks two cards. Therefore there is a number n between 1 and 9 such that the following holds.

- The first student only picks card 1,
- the second student only picks card 2,
- $\ldots$,
- the n-th student picks up cards n and $n + 1$,
- the $(n + 1)$-th student only picks up card $n + 2$,
- $\ldots$
- and the ninth student only picks card 10.

Each pair of consecutive cards has to be in reverse order, except for the cards n and $n + 1$. This means that the cards with $1, \ldots, n$ are in reverse order, the cards n and $n + 1$ are in the right order, and the cards $n + 1, \ldots, 10$ are in reverse order. (The pair of consecutive cards a, $a + 1$ lies in the right order if a lies left of $a + 1$, and in reverse order if a lies right of $a + 1$.) We are now going to replace every card k with card $11 - k$. If two cards were in the right order first, then they are now in reverse order; and if two cards were in reverse order first, then they are in the right order now. So we find that cards $1, \ldots, 11 - n - 1$ lie in the correct order, cards $11 - n - 1$ and $11 - n$ lie in reverse order, and cards $11 - n, \ldots, 10$ lie in the correct order. Then students will take cards as follows:

- The first student picks up cards $1, \ldots, 11 - n - 1$,
- The second student picks up cards $11 - n, \ldots, 10$.

So exactly two students will get a turn. This shows that if, in a sequence where exactly nine students get a turn, you replace every number k by $11-k$, then you get a sequence where exactly two students get a turn.

The opposite of this statement also needs a proof, and we proceed in a similar way. Suppose two students get a turn. Then there is a number n between 1 and 9 such that the cards $1, \ldots, n$ lie in the correct order (those get picked by the first student), cards n and $n+1$ lie in reverse order, and cards $n+1, \ldots, 10$ lie in the correct order (those get picked by the second student). Again we are going to replace every card k with card $11-k$. Then we get a sequence where the cards $1, \ldots, 11-n-1$ lie in reverse order, cards $11-n-1$ and $11-n$ lie in the correct order, and cards $11-n, \ldots, 10$ lie in reverse order. We already saw that for such a sequence, exactly nine students get a turn. This shows that if, in a sequence where exactly two students get a turn, you replace every number k by $11-k$, then you get a sequence where exactly nine students get a turn.

We conclude that for every sequence of cards in which exactly nine students take their turn, there is a sequence in which exactly two students take their turn, and vice versa. It follows that $A = B$.

Another way to prove this is by reversing the sequence of cards.