Final round Dutch Mathematical Olympiad

Version klas 4 and below

Friday 15 September 2023
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. A number is called *nillless* if it is integer and positive and contains no zeros. You can make a positive integer *nillless* by simply omitting the zeros. We denote this with square brackets, for example $[2050] = 25$ and $[13] = 13$. The following is known about the two numbers a and b:

- a and b are nillless,
- $1 < a < b < 100$,
- $[a \cdot b] = 2$.

Which pairs (a, b) satisfy these three requirements?

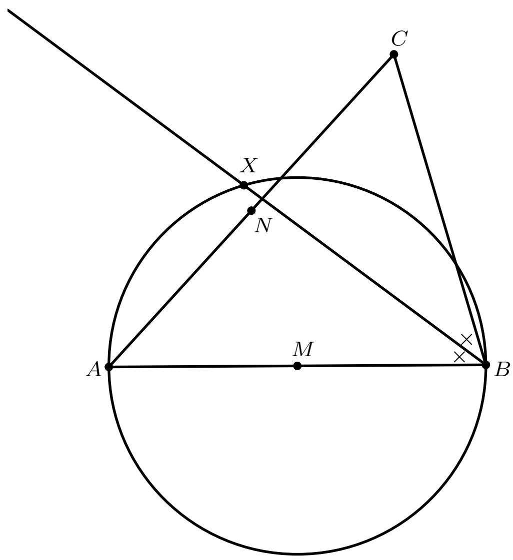
2. In a room there are 2023 vases numbered from 1 to 2023. In each vase we want to put a note with a positive integer from $1, 2, \ldots, 2023$ on it. The numbers on the notes do not necessarily have to be distinct. The following should now apply to each vase. Look at the note inside the vase, find the (not necessarily different) vase with the number written on the note, and look at the note inside this vase. Then the average of the numbers on the two notes must be exactly equal to the number of the first selected vase. For example, if we put a note with the number 5 in vase 13, then vase 5 should contain a note with the number 21 on it: after all, the average of 5 and 21 is 13.
Determine all possible ways to provide each vase with a note.

3. Felix chooses a positive integer as the starting number and writes it on the board. He then repeats the next step: he replaces the number n on the board by $\frac{1}{2}n$ if n is even and by $n^2 + 3$ if n is odd.

(a) Prove that when Felix starts with an odd number, the next two numbers he writes down are even.

(b) For how many choices of starting numbers below 2023 will Felix never write a number of more than four digits on the board?

4. In acute-angled triangle ABC with $|BC| < |BA|$, point M is the midpoint of AB and N is the midpoint of AC. The circle with diameter AB intersects the bisector of $\angle B$ in two points: B and X.
Prove that M, N, and X lie on the same line.



5. A maths teacher has 10 cards with the numbers 1 to 10 on them, one number per card. She places these cards in some order in a line next to each other on the table. The students come to the table, one at a time. The student whose turn it is goes once through the line of cards from left to right and removes every card she encounters that is (at that moment) the lowest card on the table. This continues till all cards are removed from the table. For example, if the line is in order $3, 1, 4, 5, 8, 6, 9, 10, 2, 7$ from left to right, the first student takes cards 1 and 2. Then the second student comes who, in our example, takes the cards $3, 4, 5, 6$, and 7. The third student then takes the cards $8, 9$, and 10.
Let A be the number of sequences of cards that the teacher can choose so that exactly nine students get a turn to pick cards. Let B be the number of sequences of cards that the teacher can choose so that exactly two students get a turn to pick cards.
Prove that $A = B$.