Final round
Dutch Mathematical Olympiad

Friday 16 September 2022

Solutions

1. Version for klas 5 & klas 4 and below

(a) Suppose n is divisor primary. Then n cannot have an odd divisor $d \geqslant 5$. Indeed, for such a divisor, both $d-1$ and $d+1$ are even. Because $d-1>2$, these are both composite numbers and that would contradict the fact that n is divisor primary. The odd divisors 1 and 3 can occur, because the integer 3 itself is divisor primary.

(b) Because of the unique factorisation in primes, the integer n can now only have some factors 2 and at most one factor 3. The number $2^6 = 64$ and all its multiples are not divisor primary, because both $63 = 7 \cdot 9$ and $65 = 5 \cdot 13$ are not prime. Hence, a divisor primary number has at most five factors 2. Therefore, the largest possible number that could still be divisor primary is $3 \cdot 2^5 = 96$.

We now check that 96 is indeed divisor primary: its divisors are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, and 96, and these numbers are next to 2, 3, 2, 3, 5, 7, 11, 17, 23, 31, 47, and 97, which are all prime. Therefore, the largest divisor primary number is 96.

1. Version for klas 6

See the solutions above. We investigate in part (a) which odd numbers can occur as the divisor of a divisor primary number. Then we determine in part (b) the largest divisor primary number.

2. We solve this problem in two steps. First we will show that the smallest possible integral average of a centenary set is 14, and then we will show that we can obtain all integers greater than or equal to 14, but smaller than 100, as the average of a centenary set.

If you decrease one of the numbers (unequal to 100) in a centenary set, the average becomes smaller. Also if you add a number that is smaller than the current average, the average becomes smaller. To find the centenary set with the smallest possible average, we can start with $1, 100$ and keep adjoining numbers that are as small as possible, until the next number that we would add is greater than the current average. In this way, we find the set with the numbers 1 to 13 and 100 with average $\frac{1}{14} \cdot (1+2+\ldots+13+100) = \frac{191}{14} = 13\frac{9}{14}$. Adding 14 would increase the average, and removing 13 (or more numbers) would increase the average as well. We conclude that the average of a centenary set must be at least 14 when it is required to be an integer.

Therefore, the smallest integer which could be the average of a centenary set is 14, which could for example be realised using the following centenary set:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 100\}.$$

Now we still have to show that all integers greater than 14 (and smaller than 100) can indeed be the average of a centenary set. We start with the centenary set above with average 14. Each time you add 14 to one of the numbers in this centenary set, the average increases by 1. Apply this addition from right to left, first adding 14 to 18 (the average becoming 15), then adding 14 to 12 (the average becoming 16), then adding 14 to 11, et cetera. Then you end up with the centenary set $\{15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 32, 100\}$ with average 27, and you realised all values from 14 to 27 as an average. Because we started adding 14 to the second largest number in the set, this sequence of numbers remains increasing during the whole process, and therefore consists of 14 distinct numbers the whole time, and hence the numbers indeed form a centenary set.

We can continue this process by first adding 14 to 32, then 14 to 26 et cetera, and then we get a centenary set whose average is 40. Repeating this one more time, we finally end up with the set

$\{43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 60, 100\}$ with average 53. Moreover, we can obtain 54 as the average of the centenary set $\{8, 100\}$, 55 as the average of $\{10, 100\}$, and so on until 99, which we obtain as the average of $\{98, 100\}$. This shows that all values from 14 to 99 can be obtained.

3. (a) The sequence starts as follows.

$$a_1 = \tfrac{10}{11}, \quad a_2 = \tfrac{12}{14} = \tfrac{6}{7}, \quad a_3 = \tfrac{8}{10} = \tfrac{4}{5}, \quad a_4 = \tfrac{6}{8} = \tfrac{3}{4}, \quad a_5 = \tfrac{5}{7}, \quad a_6 = \tfrac{7}{10}, \quad a_7 = \tfrac{9}{13}$$

It seems that the last simplification occurred at a_4. With induction to n, we will prove that there is no simplification for all $n \geqslant 5$. At the same time, we will prove that $a_n = \frac{1+2(n-3)}{1+3(n-3)}$ for all $n \geqslant 5$.

For $n = 5$, the statement is true, because $a_5 = \frac{5}{7} = \frac{1+2(5-3)}{1+3(5-3)}$ and this fraction $\frac{5}{7}$ cannot be simplified further. Now suppose the statement is true for $n = k-1$. Consider $n = k$. Because there has been no simplification for a_{k-1}, the numerator of a_{k-1} equals $1+2(k-4)$ and the denominator equals $1+3(k-4)$. Then the number a_k is defined as $\frac{1+2(k-4)+2}{1+3(k-4)+3} = \frac{1+2(k-3)}{1+3(k-3)}$. We will argue by contradiction that there is no simplification here. Namely, suppose there is an integer $d > 1$ such that both $1+2(k-3)$ and $1+3(k-3)$ are divisible by d. In particular, $3 \cdot (1+2(k-3)) - 2 \cdot (1+3(k-3)) = 1$ will also be divisible by d. This gives a contradiction, and the proof by induction is finished.

(b) We will show that there must be a simplification at some point. Indeed, suppose there is no simplification. Just like in part (a), we can show by induction that $a_n = \frac{97+2n}{97+3n}$. In particular, we see that a_{97} is not a simplified fraction, because both the numerator and denominator are divisible by 97, and that is a contradiction.

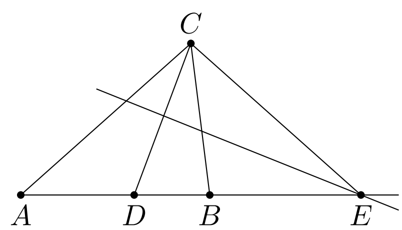
(c) You can use $c = 7$ or $c = 27$, for example. Then we get the sequences

$$\tfrac{7}{8}, \quad \tfrac{9}{11}, \quad \tfrac{11}{14}, \quad \tfrac{13}{17}, \quad \tfrac{15}{20} = \tfrac{3}{4} \qquad \text{and} \qquad \tfrac{27}{28}, \quad \tfrac{29}{31}, \quad \tfrac{31}{34}, \quad \tfrac{33}{37}, \quad \tfrac{35}{40} = \tfrac{7}{8}.$$

4. Version for klas 4 and below

(a) In triangle $\triangle ADC$, the sum of the angles is $180°$, hence

$$\angle BAC = \angle DAC = 180° - \angle ADC - \angle ACD.$$



Because CD is the angle bisector of $\angle ACB$, we have $\angle ACD = \angle DCB$ and hence the equality above can be rewritten as

$$\angle BAC = 180° - \angle ADC - \angle DCB.$$

Now we use that $\angle ADB$ is a straight angle, hence $\angle EDC = 180° - \angle ADC$. Substituting this yields

$$\angle BAC = \angle EDC - \angle DCB.$$

Because E lies on the perpendicular bisector of CD, we have $\angle EDC = \angle ECD$, and the equality becomes

$$\angle BAC = \angle ECD - \angle DCB.$$

Finally, we also see in the picture that $\angle ECD - \angle DCB = \angle BCE$, and hence

$$\angle BAC = \angle BCE.$$

(b) Triangles $\triangle ACE$ and $\triangle CBE$ are similar, because $\angle AEC = \angle CEB$ (same angle) and in part (a) we proved that $\angle BAC = \angle BCE$ and hence $\angle CAE = \angle BCE$. This yields

$$\frac{|AE|}{|CE|} = \frac{|CE|}{|BE|}.$$

(c) Using the fact that $|BE| = 4$, we compute

$$|AE| = |AB| + |BE| = 5 + 4 = 9.$$

Substituting this in the ratios above, we obtain

$$\frac{9}{|CE|} = \frac{|CE|}{4},$$

hence $|CE|^2 = 36$ and $|CE| = 6$. Because the perpendicular bisector of CD passes through E, we have $|CE| = |DE|$. This yields

$$6 = |CE| = |DE| = |DB| + |BE| = |DB| + 4$$

and hence $|DB| = 2$. Therefore, we conclude that

$$|AD| = |AB| - |BD| = 5 - 2 = 3 \quad \text{and} \quad |ED| = 6.$$

We obtain that $2|AD| = 2 \cdot 3 = 6 = |ED|$.

4. Version for klas 5 & klas 6

See the solutions above. Part (a) of klas 4 and below is equal to part (a) for klas 5 & klas 6; parts (b) and (c) of klas 4 and below together form the solution to part (b) for klas 5 & klas 6.

5. We will show that the maximum number of distinct distances is 7. First we prove that the number of distinct distances cannot be more than 7, then we will show that there is a sequence of blocks with 7 distances.

The possible distances between two blocks in the sequence are the numbers 1 to 8. Therefore, there can certainly be no more than 8 distinct distances. We will show that there is always at least one distance that does not occur.

If in a sequence the distances 8 and 7 do not both occur, we are done. Therefore, suppose we have a sequence in which these two distances do both occur. The distance 8 can only occur between the very first and the very last block, so these should have the same letter on them, say A. The distance 7 can only occur between the first and the eighth (second last) block, or between the second and the last block. Because both outer blocks have an A, the second or eighth block must also have an A. Then the sequence of blocks is AAxxxxxxA (or the other way around: AxxxxxxAA), where on the place of x are blocks with a B or C. Now we see that the distance 6 cannot occur anymore: the distances between the blocks with A are 1, 7, and 8, and the distances between the blocks with B and the blocks with C are at most 5. Also in this case, there is at least one distance that does not occur.

We conclude that there is always one of the possible distances 1, 2, 3, 4, 5, 6, 7, 8 that does not occur. Hence, the number of distinct distances cannot be more than 7.

An example of a sequence of blocks where 7 distinct distances occur, is ABBCACCBA, with distances 4, 4, 8; 1, 5, 6; 1, 2, 3 (only the distance 7 is missing). So the maximal number of distinct distances is equal to 7.