Final round
Dutch Mathematical Olympiad

Version klas 5

Friday 16 September 2022
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. A positive integer n is called *divisor primary* if for every positive divisor d of n at least one of the numbers $d-1$ and $d+1$ is prime. For example, 8 is divisor primary, because its positive divisors 1, 2, 4, and 8 each differ by 1 from a prime number (2, 3, 5, and 7, respectively), while 9 is not divisor primary, because the divisor 9 does not differ by 1 from a prime number (both 8 and 10 are composite).

(a) Which odd numbers can occur as the divisor of a divisor primary number?

(b) Determine the largest divisor primary number.

2. A set consisting of at least two distinct positive integers is called *centenary* if its greatest element is 100. We will consider the average of all numbers in a centenary set, which we will call the average of the set. For example, the average of the centenary set $\{1, 2, 20, 100\}$ is $\frac{123}{4}$ and the average of the centenary set $\{74, 90, 100\}$ is 88.
Determine all integers that can occur as the average of a centenary set.

3. Given a positive integer c, we construct a sequence of fractions $a_1, a_2, a_3, \ldots$ as follows:

- $a_1 = \frac{c}{c+1}$;
- to get a_n, we take a_{n-1} (in its most simplified form, with both the numerator and denominator chosen to be positive) and we add 2 to the numerator and 3 to the denominator. Then we simplify the result again as much as possible, with positive numerator and denominator.

For example, if we take $c = 20$, then $a_1 = \frac{20}{21}$ and $a_2 = \frac{22}{24} = \frac{11}{12}$. Then we find that $a_3 = \frac{13}{15}$ (which is already simplified) and $a_4 = \frac{15}{18} = \frac{5}{6}$.

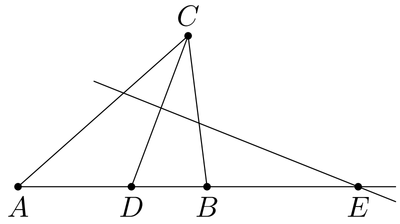
(a) Let $c = 10$, hence $a_1 = \frac{10}{11}$. Determine the largest n for which a simplification is needed in the construction of a_n.

(b) Let $c = 99$, hence $a_1 = \frac{99}{100}$. Determine whether a simplification is needed somewhere in the sequence.

(c) Find a value of c for which in the first step of the construction of a_5 (before simplification) the numerator and denominator are divisible by 5.

PLEASE CONTINUE ON THE OTHER SIDE

4. In triangle ABC, the point D lies on segment AB such that CD is the angle bisector of angle C. The perpendicular bisector of segment CD intersects the line AB in E. Suppose that $|BE| = 4$ and $|AB| = 5$.



(a) Prove that $\angle BAC = \angle BCE$.

(b) Prove that $2|AD| = |ED|$.

5. Kira has 3 blocks with the letter A, 3 blocks with the letter B, and 3 blocks with the letter C. She puts these 9 blocks in a sequence. She wants to have as many distinct distances between blocks with the same letter as possible. For example, in the sequence ABCAABCBC the blocks with the letter A have distances 1, 3, and 4 between one another, the blocks with the letter B have distances 2, 4, and 6 between one another, and the blocks with the letter C have distances 2, 4, and 6 between one another. Altogether, we got distances of 1, 2, 3, 4, and 6; these are 5 distinct distances.
What is the maximum number of distinct distances that can occur?