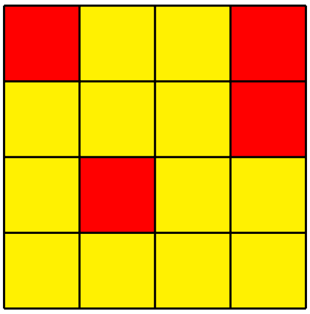
Final round
Dutch Mathematical Olympiad

Version klas 6

Friday 17 September 2021
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. Niek has 16 square cards that are yellow on one side and red on the other. He puts down the cards to form a 4×4-square. Some of the cards show their yellow side and some show their red side. For a colour pattern he calculates the *monochromaticity* as follows. For every pair of adjacent cards that share a side he counts $+1$ or -1 according to the following rule: $+1$ if the adjacent cards show the same colour, and -1 if the adjacent cards show different colours. Adding this all together gives the monochromaticity (which might be negative). For example, if he lays down the cards as below, there are 15 pairs of adjacent cards showing the same colour, and 9 such pairs showing different colours.



The monochromaticity of this pattern is thus $15 \cdot (+1) + 9 \cdot (-1) = 6$. Niek investigates all possible colour patterns and makes a list of all possible numbers that appear at least once as a value of the monochromaticity. That is, Niek makes a list with all numbers such that there exists a colour pattern that has this number as its monochromaticity.

(a) What are the three largest numbers on his list?
(Explain your answer. If your answer is, for example, 12, 9 and 6, then you have to show that these numbers do in fact appear on the list by giving a colouring for each of these numbers, and furthermore prove that the numbers 7, 8, 10, 11 and all numbers bigger than 12 do not appear.)

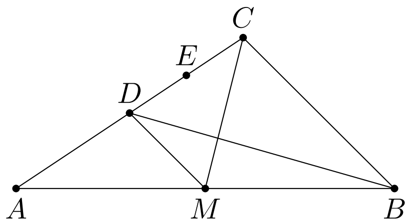
(b) What are the three smallest (most negative) numbers on his list?

(c) What is the smallest positive number (so, greater than 0) on his list?

2. We consider sports tournaments with $n \geqslant 4$ participating teams and where every pair of teams plays against one another at most one time. We call such a tournament *balanced* if any four participating teams play exactly three matches between themselves. So, not all teams play against one another.
Determine the largest value of n for which a balanced tournament with n teams exists.

3. A frog jumps around on the grid points in the plane, from one grid point to another. The frog starts at the point $(0, 0)$. Then it makes, successively, a jump of one step horizontally, a jump of 2 stepsvertically, a jump of 3 steps horizontally, a jump of 4 steps vertically, et cetera. Determine all $n > 0$ such that the frog can be back in $(0, 0)$ after n jumps.

4. In triangle ABC we have $\angle ACB = 90°$. The point M is the middle of AB. The line through M parallel to BC intersects AC in D. The midpoint of line segment CD is E. The lines BD and CM are perpendicular.



Be aware: the figure is not drawn to scale.

(a) Prove that triangles CME and ABD are similar.

(b) Prove that EM and AB are perpendicular.

5. We consider an integer $n > 1$ with the following property: for every positive divisor d of n we have that $d + 1$ is a divisior of $n + 1$. Prove that n is a prime number.