Final round
Dutch Mathematical Olympiad

Friday 11 September 2020

Solutions

1. (a) The smallest possible number of circled numbers is 3. Fewer than 3 is not possible since in each row at least one number is circled (and these are three different numbers).

On the right, a median table is shown in which only 3 numbers are circled. In the rows, the numbers $7, 5, 3$ are circled, in the columns the numbers $3, 5, 7$, and on the diagonals the numbers 5 and 5. Together, these are three different numbers: 3, 5, and 7.

4	9	7
2	5	8
3	1	6

(b) The largest possible number of circled numbers is 7. More than 7 is not possible, since the numbers 9 and 1 are never circled, hence no more than $9 - 2 = 7$ numbers are circled.

On the right, a median table is shown in which 7 numbers are circled. In the rows, the numbers $2, 6, 8$ are circled, in the columns the numbers $7, 5, 3$, and on the diagonals the numbers 4 and 5. Together, these are the numbers $2, 3, 4, 5, 6, 7, 8$.

4	1	2
7	5	6
8	9	3

2. Version for klas 5 & klas 4 and below

(a) If $a_1 = -3$, we have $a_2 = \frac{a_1+a_1}{a_1+1} = \frac{-6}{-2} = 3$. Next, we find $a_3 = \frac{a_2+a_1}{a_2+1} = \frac{0}{4} = 0$. Then, we have $a_4 = \frac{a_3+a_1}{a_3+1} = \frac{-3}{1} = -3$. We see that $a_4 = a_1$. Since a_{n+1} only depends on a_n and a_1, we see that $a_5 = a_2 = 3$, $a_6 = a_3 = 0$, $a_7 = a_4 = -3$, et cetera. In other words: the sequence is periodic with period 3, and we see that

$$a_{2020} = a_{2017} = a_{2014} = \cdots = a_4 = a_1 = -3.$$

(b) We will prove the statement by induction on n. We start with the induction basis $n = 2$. We calculate $a_2 = \frac{a_1+a_1}{a_1+1} = \frac{4}{3}$ and see that indeed $\frac{4}{3} \leqslant a_n \leqslant \frac{3}{2}$ holds.

Now suppose that for certain $n = k \geqslant 2$ we have proven that $\frac{4}{3} \leqslant a_n \leqslant \frac{3}{2}$. We will show that these inequalities hold for $n = k + 1$ as well.

We first observe that

$$a_{k+1} = \frac{a_k + 2}{a_k + 1} = 1 + \frac{1}{a_k + 1}.$$

From the induction hypothesis ($\frac{4}{3} \leqslant a_k \leqslant \frac{3}{2}$) it follows that $\frac{7}{3} \leqslant a_k + 1 \leqslant \frac{5}{2}$, and hence that

$$\frac{2}{5} \leqslant \frac{1}{a_k + 1} \leqslant \frac{3}{7}.$$

By adding 1 to all parts of these inequalities, we find

$$\frac{7}{5} \leqslant 1 + \frac{1}{a_k + 1} = a_{k+1} \leqslant \frac{10}{7}.$$

Since $\frac{4}{3} \leqslant \frac{7}{5}$ and $\frac{10}{7} \leqslant \frac{3}{2}$, we see that $\frac{4}{3} \leqslant a_{k+1} \leqslant \frac{3}{2}$, completing the induction step.

2. Version for klas 6

(a) First, we determine for what starting values $a_1 > 0$ the inequalities $\frac{4}{3} \leqslant a_2 \leqslant \frac{3}{2}$ hold. Then, we will prove that for those starting values, the inequalities $\frac{4}{3} \leqslant a_n \leqslant \frac{3}{2}$ are also valid for all $n \geqslant 2$.

First, we observe that $a_2 = \frac{a_1+2}{a_1+1}$ and that the denominator, a_1+1, is positive (since $a_1 > 0$). The inequality

$$\frac{4}{3} \leqslant a_2 = \frac{a_1+2}{a_1+1} \leqslant \frac{3}{2},$$

is therefore equivalent to the inequality

$$\tfrac{4}{3}(a_1+1) \leqslant a_1+2 \leqslant \tfrac{3}{2}(a_1+1),$$

as we can multiply all parts in the inequality by the positive number a_1+1. Subtracting a_1+2 from all parts of the inequality, we see that this is equivalent to

$$\tfrac{1}{3}a_1 - \tfrac{2}{3} \leqslant 0 \leqslant \tfrac{1}{2}a_1 - \tfrac{1}{2}.$$

We therefore need to have $\frac{1}{3}a_1 \leqslant \frac{2}{3}$ (i.e. $a_1 \leqslant 2$), and $\frac{1}{2} \leqslant \frac{1}{2}a_1$ (i.e. $1 \leqslant a_1$). The starting value a_1 must therefore satisfy $1 \leqslant a_1 \leqslant 2$.

Now suppose that $1 \leqslant a_1 \leqslant 2$, so that a_2 satisfies $\frac{4}{3} \leqslant a_2 \leqslant \frac{3}{2}$. Looking at a_3, we see that $a_3 = \frac{a_2+2}{a_2+1}$. That is the same expression as for a_2, only with a_1 replaced by a_2. Since a_2 also satisfies $1 \leqslant a_2 \leqslant 2$, the same argument now shows that $\frac{4}{3} \leqslant a_3 \leqslant \frac{3}{2}$.

We can repeat the same argument to show this for a_4, a_5, etcetera. Hence, we find that $\frac{4}{3} \leqslant a_n \leqslant \frac{3}{2}$ holds for all $n \geqslant 2$. The formal proof is done using induction: the induction basis $n=2$ has been shown above. For the induction step, see the solution of part (b) of the version for klas 5 & klas 4 and below. The result is that all inequalities hold if and only if $1 \leqslant a_1 \leqslant 2$.

(b) Let's start by computing the first few numbers of the sequence in terms of a_1. We see that

$$a_2 = \frac{a_1-3}{a_1+1} \quad \text{and} \quad a_3 = \frac{a_2-3}{a_2+1} = \frac{\frac{a_1-3}{a_1+1}-3}{\frac{a_1-3}{a_1+1}+1} = \frac{a_1-3-3(a_1+1)}{a_1-3+(a_1+1)} = \frac{-2a_1-6}{2a_1-2} = \frac{-a_1-3}{a_1-1}.$$

Here, it is important that we do not divide by zero, that is, $a_1 \neq -1$ and $a_2 \neq -1$. The first inequality follows directly from the assumption. For the second inequality we consider when $a_2 = -1$ holds. This is the case if and only if $a_1 - 3 = -(a_1+1)$, if and only if $a_1 = 1$. Since we assumed that $a_1 \neq 1$, we see that $a_2 \neq -1$. The next number in the sequence is

$$a_4 = \frac{a_3-3}{a_3+1} = \frac{\frac{-a_1-3}{a_1-1}-3}{\frac{-a_1-3}{a_1-1}+1} = \frac{-a_1-3-3(a_1-1)}{-a_1-3+(a_1-1)} = \frac{-4a_1}{-4} = a_1.$$

Again, we are not dividing by zero since $a_3 = -1$ only holds when $-a_1-3 = -a_1+1$, which is never the case.

We see that $a_4 = a_1$. Since a_{n+1} only depends on a_n, we see that $a_5 = a_2$, $a_6 = a_3$, $a_7 = a_4$, et cetera. In other words: the sequence is periodic with period 3, and we see that

$$a_{2020} = a_{2017} = a_{2014} = \cdots = a_4 = a_1.$$

To conclude: indeed we have $a_{2020} = a_1$ for all starting values a_1 unequal to 1 and -1.

3. Version for klas 4 & below

(a) Since AD and BC are parallel, we have (F angles): $\angle CMN = \angle DAM = \frac{1}{2}\angle DAB$. Since DN and AB are parallel, we have (Z angles): $\angle CNM = \angle NAB = \frac{1}{2}\angle DAB$. It follows that $\angle CMN = \angle CNM$, so triangle CMN is isosceles with apex C. We obtain $|CM| = |CN|$. Line segments OC, ON, and OM are radii of the same circle, and therefore of equal length. Triangles OCM and OCN are therefore congruent (three pairs of equal sides).

(b) To show that $\angle OBC = \angle ODC$, we will show that triangles OBC and ODN are congruent. We will do this using the ZHZ-criterion. We will show that $\angle OND = \angle OCB$, and $|ON| = |OC|$, and $|DN| = |BC|$.

The equality $|ON| = |OC|$ follows since ON and OC are radii of the same circle. In part (a), we saw triangles OCM and OCN are congruent. Furthermore, these two triangles are isosceles ($|OC| = |OM|$ and $|OC| = |ON|$). Hence, the four base angles $\angle ONC$, $\angle OCN$, $\angle OMC$, and $\angle OCM$ are equal. We see that $\angle OND = \angle OCB$. The only thing we still need to show is that $|DN| = |BC|$.

Observe that $\angle BMA = \angle DAM$ (Z angles) and $\angle DAM = \angle MAB$ (as AM is the angular bisector of A). We find that $\angle BMA = \angle MAB$. Triangle AMB is therefore isosceles and we have $|AB| = |BM|$. We previously saw that $|CM| = |CN|$, and we also have $|AB| = |CD|$ as $ABCD$ is a parallelogram. We therefore obtain

$$|DN| = |CD| + |CN| = |AB| + |CM| = |BM| + |CM| = |BC|,$$

which concludes the proof.

3. Version for klas 5 & klas 6

As an intermediate step, we first show that triangles OCM and OCN are congruent. The solution for this step can be found in part (a) of the version for klas 4 & below. The remainder of the solution is part (b) of that version.

4. Version for klas 4 & below

We may assume that $a \geqslant 0$. If (x, y) is a solution, then $(-x, -y)$ is a solution as well. Therefore, we may for now assume that $x + y \geqslant 0$ and, at the end, add for each solution (x, y) the pair $(-x, -y)$ to the list of solutions.

We see that

$$p = (x + y)^2 - a^2 = (x + y + a)(x + y - a).$$

It follows that $x + y + a$ is non-zero, and hence positive. Then $x + y - a$ must be positive as well. The prime number p can be written as a product of two positive integers in only two ways: $1 \cdot p$ and $p \cdot 1$. Since $x + y + a \geqslant x + y - a$, we must have $x + y + a = p$ and $x + y - a = 1$. Adding these two equations, we obtain $2x + 2y = p + 1$. We also know that $x^2 + y^2 + 1 = p + 1$, so we obtain $2x + 2y = x^2 + y^2 + 1$. By bringing all terms to the right-hand side and adding 1 to both sides, we find

$$1 = x^2 + y^2 - 2x - 2y + 2 = (x - 1)^2 + (y - 1)^2.$$

We now have two perfect squares that add up to 1. This means that one of the squares is equal to 1 and the other is equal to 0. Hence, $(x - 1)^2 = 0$ and $(y - 1)^2 = 1$, or $(x - 1)^2 = 1$ and $(y - 1)^2 = 0$. In the first case we have $x - 1 = 0$ and $y - 1 = \pm 1$, so (x, y) is equal to $(1, 2)$ or $(1, 0)$. In the second case we have $x - 1 = \pm 1$ and $y - 1 = 0$, so (x, y) is equal to $(2, 1)$ or $(0, 1)$. Since $0^2 + 1^2 = 1$ is not a prime number, $(1, 0)$ and $(0, 1)$ are eliminated as candidate solutions.

We now check whether $(1, 2)$ and $(2, 1)$ are solutions. In both cases we find $p = x^2+y^2 = 1^2+2^2 = 5$, which is a prime number as required. Taking $a = 2$, we have $(x + y)^2 - a^2 = 3^2 - 2^2 = 5$, which is indeed equal to p.

Adding the solutions obtained by replacing (x, y) by $(-x, -y)$, we find a total of four solutions, namely

$$(1, 2), (2, 1), (-1, -2), (-2, -1).$$

4. Version for klas 5 & klas 6

We have $2xy = a^2$ for some nonnegative integer a, and $x^2 + y^2 = p$ for some prime number p.

Since a prime number is never a perfect square, we see that $x, y \neq 0$. Since $2xy$ is a perfect square, it follows that x and y must both be positive, or both be negative. If (x, y) is a solution, then so is $(-x, -y)$. Therefore, we may for now assume that x and y are positive, and at the end, add for each solution (x, y) the pair $(-x, -y)$ to the list of solutions.

Combining $2xy = a^2$ and $x^2 + y^2 = p$ yields $(x + y)^2 = x^2 + y^2 + 2xy = p + a^2$. By bringing a^2 to the other side, we find

$$p = (x + y)^2 - a^2 = (x + y + a)(x + y - a).$$

Since $x+y+a$ is positive, also $x+y-a$ must be positive. The prime number p can be written as a product of two positive integers in only two ways: $1 \cdot p$ and $p \cdot 1$. Since $x+y+a \geqslant x+y-a$, we obtain $x+y+a=p$ and $x+y-a=1$.

Adding these two equations, we get $2x+2y=p+1$. We also know that $x^2+y^2+1=p+1$, so $2x+2y=x^2+y^2+1$. By bringing all terms to the right-hand side and adding 1 to both sides, we obtain

$$1=x^2+y^2-2x-2y+2=(x-1)^2+(y-1)^2.$$

We now have two perfect squares that add up to 1. This implies that one of the squares is 0 and the other is 1. So $(x-1)^2=0$ and $(y-1)^2=1$, or $(x-1)^2=1$ and $(y-1)^2=0$. As x and y are positive, we find two possible solutions: $x=1$ and $y=2$, or $x=2$ and $y=1$. In both cases $2xy=4$ is a perfect square and $x^2+y^2=5$ is a prime number. It follows that both are indeed solutions.

Adding the solutions obtained by replacing (x,y) by $(-x,-y)$, we obtain a total of four solutions (x,y), namely

$$(1,2),(2,1),(-1,-2),(-2,-1).$$

5. Suppose that on a given day, Sabine is left with n^2 shells, where $n>1$. Then the next day, she will give n shells to her sister and will be left with n^2-n shells. This is more than $(n-1)^2$, since

$$(n-1)^2=n^2-2n+1=(n^2-n)-(n-1)<n^2-n$$

as $n>1$. The next day, she therefore gives $n-1$ shells to her sister and is left with $n^2-n-(n-1)=(n-1)^2$ shells, again a perfect square. We see that the numbers of shells that Sabine is left with are alternately a perfect square and a number that is not a perfect square.

Let d be the first day that Sabine is left with a number of shells that is a perfect square, say n^2 shells. Then days $d+2, d+4, \ldots, d+18$ are the second to tenth day that the remaining number of shells is a perfect square (namely $(n-1)^2,(n-2)^2,\ldots,(n-9)^2$ shells). We conclude that $d+18=28$, and hence $d=10$.

On day 26 the number of remaining shells is at least 1000, but on days 27 and 28 this number is less than 1000. We see that $(n-9)^2<1000 \leqslant (n-8)^2$. As $31^2<1000 \leqslant 32^2$, we see that $n-8=32$, and hence $n=40$. We find that day 10 is the first day that the number of remaining shells is a perfect square, and that this number is 40^2.

In the remainder of the proof, we will use the following observation.

Observation. On any day, starting with more shells, means that Sabine will have more (or just as many) shells left after giving shells to her sister.

Indeed, suppose that Sabine starts the day with x shells, say $n^2 \leqslant x<(n+1)^2$. After giving away shells, she will be left with $x-n$ shells. If she had started with $x+1$ shells instead of x, she would have been left with $x+1-n>x-n$ or $x+1-(n+1)=x-n$ shells.

Let x be the number of shells remaining on day 8. The obvious guess $x=41^2=1681$ is incorrect as x cannot be a perfect square. We therefore try $x=41^2-2$, $x=41^2-1$, and $x=41^2+1$. The table shows the number of shells remaining on day 8, 9, and 10.

day 8	day 9	day 10
$41^2-2=1679$	$1679-40=1639$	$1639-40=1599$
$41^2-1=1680$	$1680-40=1640$	$1640-40=1600$
$41^2+1=1682$	$1682-41=1641$	$1641-40=1601$

We see that the case $x=1679$ is ruled out because it would imply that fewer than $40^2=1600$ shells are left on day 10. By the above observation, this also rules out the case $x<1679$. The case $x = 1682$ is ruled out because it would imply that more than 40^2 shells will be left on day 10. Hence, also $x > 1682$ is ruled out. The number of shells left on day 8 must therefore be $41^2 - 1$.

To follow the pattern back in time, we consider the case that the number of remaining shells is just shy of a perfect square. Suppose that on a given day the number of remaining shells is $n^2 - a$, where $1 \leqslant a < n$. Then the following day, the number of remaining shells is $n^2 - a - (n-1)$. Since $a < n$, we have $n^2 - a - (n-1) > n^2 - n - (n-1) = (n-1)^2$. The day after that, the number of remaining shells must therefore be $n^2 - a - (n-1) - (n-1) = (n-1)^2 - (a-1)$.

So if Sabine originally had $45^2 - 5$ shells, then the number of remaining shells on days 2, 4, 6, and 8 are $44^2 - 4$, $43^2 - 3$, $42^2 - 2$, and $41^2 - 1$, respectively. This gives us a solution.

If Sabine originally had $45^2 - 4$ shells, then she would be left with too many shells on day 8, namely $41^2 - 0$. The original number of shells could therefore not have been $45^2 - 4$ or more.

If Sabine originally had $45^2 - 6$ shells, then she would be left with too few shells on day 8, namely $41^2 - 2$. The original number of shells could therefore not have been $45^2 - 6$ or fewer.

We conclude that the only possibility is that Sabine started with a collection of $45^2 - 5 = 2020$ shells.