# Final round
# Dutch Mathematical Olympiad

Friday 13 September 2019

## Solutions

**1. Version for klas 4 & below**

(a) For $a = 4$, an example of such a number is 126734895. For $a = 5$, an example is the number 549832761. (There are other solutions as well.)

(b) We will show that for $a = 3, 6, 7, 8, 9$ there is no complete number with a difference number equal to $1a1a1a1a$. It then immediately follows that there is also no complete number $N$ with difference number equal to $a1a1a1a1$ (otherwise, we could write the digits of $N$ in reverse order and obtain a complete number with difference number $1a1a1a1a$).

For $a$ equal to 6, 7, 8, and 9, no such number $N$ exists for the following reason. For the digits 4, 5, and 6, there is no digit that differs by $a$ from that digit. Since the difference number of the complete number $N$ is equal to $1a1a1a1a$, every digit of $N$, except the first, must be next to a digit that differs from it by $a$. Hence, the digits 4, 5, and 6 can only occur in the first position of $N$, which is impossible.

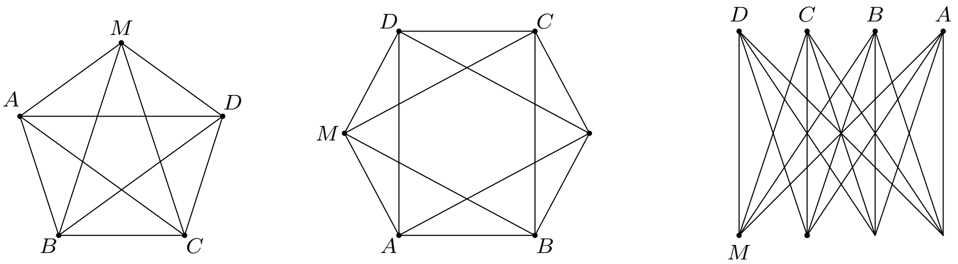
For $a = 3$ the argument is different. If we consider the digits that differ by 3, we find the triples 1–4–7, 2–5–8, and 3–6–9. If the 1 is next to the 4 in $N$, the 7 cannot be next to the 4 and so the 7 must be the first digit of $N$. If the 1 is not next to the 4, the 1 must be the first digit of $N$. In the same way, either the 2 or the 8 must be the first digit of $N$ as well. This is impossible.

**1. Version for klas 5 & klas 6**

The solutions for klas 4 & below show that this is possible for $a = 4$ and $a = 5$, but impossible for all other values of $a$.

**2.** We first consider the friends of one guest, say Marieke. We know that Marieke has exactly four friends at the party, say Aad, Bob, Carla, and Demi. The other guests (if there are any other guests) are not friends with Marieke. Hence, they cannot have any friendships among themselves and can therefore only be friends with Aad, Bob, Carla, and Demi. Since everyone has exactly four friends at the party, each of them must be friends with Aad, Bob, Carla, and Demi (and with no one else).

Since Aad also has exactly four friends (including Marieke), the group of guests that are not friends with Marieke can consist of no more than three people. If the group consists of zero, one, or three people, we have the following solutions (two guests are connected by a line if they are friends):




*Solutions with five, six, and eight guests in total.*

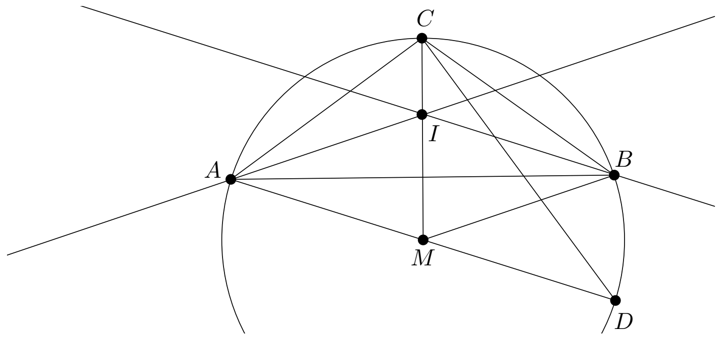
Now we will show that it is not possible for this group to consist of two people. In that case, Aad would have exactly one friend among Bob, Carla, and Demi. Assume, without loss of generality, that Aad and Bob are friends. In the same way, Carla must be friends with one of

Aad, Bob, and Demi. Since Aad and Bob already have four friends, Carla and Demi must be friends. However, since they are both not friends with Aad, this contradicts the requirement in the problem statement.

We conclude that there can be five, six, or eight guests at the party. Hence, the possible values for $n$ are 5, 6, and 8.

**3. Version for klas 4 & below**

Let $I$ be the reflection of point $M$ in the line $AB$. We define $\alpha = \angle CAI$ and $\beta = \angle CBI$. Since $AI$ is the angular bisector of $\angle CAB$, we find that $\angle IAB = \alpha$. Since $I$ is the reflection of $M$ in the line $AB$, we find that $\angle BAM = \alpha$. Triangle $AMC$ is isosceles with apex $M$, because $|AM| = |CM|$. Hence, we see that $\angle MCA = \angle CAM = 3\alpha$. In the same way, we see that $\angle IBA = \angle ABM = \beta$ and $\angle MCB = 3\beta$. The sum of the angles of triangle $ABC$ is therefore $2\alpha + (3\alpha + 3\beta) + 2\beta = 180°$. From this, we conclude that $\alpha + \beta = \frac{180°}{5} = 36°$, and hence that $\angle ACB = 3\alpha + 3\beta = 3 \cdot 36° = 108°$.




**3. Version for klas 5 & klas 6**

This solution continues where the solution for klas 4 & below finished. Please read that solution first.

Since $MAB$ is an isosceles triangle (as $|AM| = |BM|$), we see that $\alpha = \beta = 18°$. It follows from this that $\angle CAB = 2\alpha = \angle ABC$ and therefore that triangle $ACB$ is isosceles. By considering the sum of the angles in triangle $AMC$, we find that $\angle AMC = 180° - 6\alpha = 72°$. Hence we also find that $\angle CMD = 180° - \angle AMC = 108°$. We have already seen that $\angle ACB = 108°$. It follows that triangles $ACB$ and $CMD$ are both isosceles triangles with an angle of 108° at the apex. Hence, they are similar triangles. This implies that

$$\frac{|CM|}{|CD|} = \frac{|AC|}{|AB|}.$$

By multiplying by both denominators and observing that $|CM| = |AM|$, we obtain the required result.

**4. Version for klas 5 & klas 4 and below**

Note that

$$a_n = \frac{1}{n^2 + 3n + 2} = \frac{(n+2) - (n+1)}{(n+2)(n+1)} = \frac{1}{n+1} - \frac{1}{n+2}$$

for all $n \geqslant 0$. For each $m \geqslant 0$, we now find that

$$a_0 + a_1 + a_2 + \cdots + a_m = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m+1} - \frac{1}{m+2}\right).$$

In this sum all terms cancel, except the first and the last. In this way, we get that

$$a_0 + a_1 + a_2 + \cdots + a_m = \frac{1}{1} - \frac{1}{m+2} < 1.$$

*In order to find a solution, you could start by making a table containing the values of $a_m$ and $a_0+\cdots+a_m$ for $m=0,1,2,3,4,5$. Then try to find a pattern for the values of $a_0+\cdots+a_m$, for example $a_0+\cdots+a_m=\frac{m+1}{m+2}$. This can be proven directly in a clever way, like in the solution above, but it is also possible to prove this by induction on $m$.*

**4. Version for klas 6**

Note that for all $n \geqslant 0$ the number $a_n$ can be rewritten as follows:

$$\begin{aligned} a_n &= \frac{1}{F_nF_{n+2}} \\ &= \frac{F_{n+1}}{F_nF_{n+2}} \cdot \frac{1}{F_{n+1}} \\ &= \frac{F_{n+2}-F_n}{F_nF_{n+2}} \cdot \frac{1}{F_{n+1}} \\ &= \left(\frac{1}{F_n} - \frac{1}{F_{n+2}}\right) \cdot \frac{1}{F_{n+1}} \\ &= \frac{1}{F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}. \end{aligned}$$

We now get that for each $m \geqslant 0$ the sum $a_0+a_1+a_2+\cdots+a_m$ equals

$$\left(\frac{1}{F_0F_1} - \frac{1}{F_1F_2}\right) + \left(\frac{1}{F_1F_2} - \frac{1}{F_2F_3}\right) + \left(\frac{1}{F_2F_3} - \frac{1}{F_3F_4}\right) + \cdots + \left(\frac{1}{F_mF_{m+1}} - \frac{1}{F_{m+1}F_{m+2}}\right).$$

In this sum all terms cancel, except the first and last. In this way, we get that

$$a_0+a_1+a_2+\cdots+a_m = \frac{1}{F_0F_1} - \frac{1}{F_{m+1}F_{m+2}} = 1 - \frac{1}{F_{m+1}F_{m+2}} < 1.$$

*In order to find a solution, you could start by making a table containing the values of $a_m$ and $a_0+\cdots+a_m$ for $m=0,1,2,3,4,5$. Then try to find a pattern for the values of $a_0+\cdots+a_m$, for example $a_0+\cdots+a_m = 1-\frac{1}{F_{m+1}F_{m+2}}$. This can be proven directly in a clever way, like in the solution above, but it is also possible to prove this by induction on $m$.*

**5.** (a) Thomas and Nils both make 1009 moves and Nils makes the last move. Nils can make sure that the last card on the table contains a number that is *not* divisible by 3. Indeed, he could start taking cards with numbers that are divisible by 3, until all these cards are gone. Because there are only 672 such cards, he has enough turns to achieve that.

We now consider the situation before the last move of Nils. Let $k$ be the number on the last card, and let the sums of the numbers of Thomas and Nils at that very moment be $a$ and $b$. Nils has two options. If he gives away the last card, the difference between the outcomes becomes $(a+k)-b$, and if he keeps the card, the difference becomes $a-(b+k)$. Nils is able to win, unless both numbers are divisible by 3. But in that case $(a+k-b)-(a-b-k)=2k$ would also be divisible by 3. Because $k$ is not divisible by 3, the number $2k$ is also not divisible by 3 and hence Nils can win with certainty.

(b) Nils can win. We distinguish three types of cards, depending on the number on the card: type 1 (the number has remainder 1 when dividing by 3), type 2 (the number has remainder 2 when dividing by 3), and type 3 (the number is divisible by 3). Because $2019 = 3 \cdot 673$ and the card 2020 is of type 1, there are 674 cards of type 1, 673 cards of type 2, and 673 cards of type 3.

In order to win, Nils chooses a card of type 3 in his first turn (and gives it to Thomas). Then there are 674 cards of type 1 left, 673 of type 2, and 672 of type 3. In the next turns he responds to Thomas's move in the following way (as long as he is able to).

(i) If Thomas chooses a card of type 1, then Nils chooses a card of type 2 and gives it to the same person that got Thomas's card.

(ii) If Thomas chooses a card of type 2, then Nils chooses a card of type 1 and gives it to the same person that got Thomas's card.

(iii) If Thomas chooses a card of type 3, then Nils does the same (and gives the card to Thomas).

As long as Nils keeps this up, the sum of each player's cards is divisible by 3 after his turn (because a number of type 1 and a number of type 2 add up to a number which is divisible by 3).

Because the number of cards of type 3 is always *even* after Nils's turn, Nils can always execute his planned move in case (iii). Because the number of cards of type 1 is always 1 greater than that of type 2 after Nils's turn, he can also always execute his planned move in case (ii). Only at the moment when all cards of type 2 are gone and Thomas takes the last card of type 1 (case (i)), Nils cannot execute his planned move. However, in that case Nils cannot lose anymore. Indeed, after Thomas's turn the sum of the cards of one player is still divisible by 3, but the sum of the cards of the other player is not divisible by 3 anymore. Because there are only cards of type 3 left now, this will stay the same until all cards are gone. At the end, the difference between the sums of both players is not divisible by 3 and Nils wins.