Final round
Dutch Mathematical Olympiad

Friday 14 September 2018

Solutions

1. Version for klas 4 & below

First observe that a shuffle number can only contain the digits 2, 4, 6, and 8. Indeed, if we place the digits in any order, we obtain an *even* number (since it is divisible by 12) because of property (3). Since the last digit of an even number is also even, the digit must be 2, 4, 6, or 8 since it cannot be 0 because of property (1). Since we can put any of the digits in the last position, this holds for each digit of a shuffle number.

Next, observe that a shuffle number can only contain the digits 4 and 8. Indeed, suppose that we had a shuffle number containing the digit 2. We could reorder the digits so that the last digit is 2. The last two digits would then be 22, 42, 62, or 82. But then the number would not be divisible by 4 (and hence also not divisible by 12), contradicting property (3). In the same way we see that a shuffle number cannot contain digit 6 because a number ending in digits 26, 46, 66, or 86 is not divisible by 4.

A shuffle number is divisible by 3 (since it is divisible by 12), hence the sum of its digits is divisible by 3 as well. Since each digit is either 4 or 8, we have the following cases:

- 5 fours and 0 eights. Sum of the digits: 20
- 4 fours and 1 eight. Sum of the digits: **24**
- 3 fours and 2 eights. Sum of the digits: 28
- 2 fours and 3 eights. Sum of the digits: 32
- 1 four and 4 eights. Sum of the digits: **36**
- 0 fours and 5 eights. Sum of the digits: 40

The 5-digit numbers for which properties (1) and (3) hold, are precisely the numbers that have 4 fours and 1 eight, or have 1 four and 4 eights. It remains to examine which of these eight numbers are divisible by 11. For this we use the 11-criterion: a number is divisible by 11 if the *alternating sum* of the digits is divisible by 11. By alternating sum we mean that instead of adding them, we alternately add and subtract. For the eight candidate solutions we find the following alternating sums:

84444	$8-4+4-4+4=8$	48888	$4-8+8-8+8=4$
48444	$4-8+4-4+4=0$	84888	$8-4+8-8+8=12$
44844	$4-4+8-4+4=8$	88488	$8-8+4-8+8=4$
44484	$4-4+4-8+4=0$	88848	$8-8+8-4+8=12$
44448	$4-4+4-4+8=8$	88884	$8-8+8-8+4=4$

We see that there are precisely two 5-digit shuffle numbers, namely 48444 and 44484.

1. Version for klas 5 & klas 6

First observe that a shuffle number can only contain the digits 2, 4, 6, and 8. Indeed, if we place the digits in any order, we obtain an *even* number (since it is divisible by 12) because of property (3). Since the last digit of an even number is also even, the digit must be 2, 4, 6, or 8 since it cannot be 0 because of property (1). Since we can put any of the digits in the last position, this holds for each digit of a shuffle number.

Next, observe that a shuffle number can only contain the digits 4 and 8. Indeed, suppose that we had a shuffle number containing the digit 2. We could reorder the digits so that the last digit is 2. The last two digits would then be 22, 42, 62, or 82. But then the number would not be divisible by 4 (and hence also not divisible by 12), contradicting property (3). In the same way we see that a shuffle number cannot contain digit 6 because a number ending in digits 26, 46, 66, or 86 is not divisible by 4.

A shuffle number is divisible by 3 (since it is divisible by 12), hence the sum of its digits is divisible by 3 as well. If our 10-digit number has k eights and $10 - k$ fours, then the sum of its digits is equal to $8k + 4(10 - k) = 40 + 4k = 36 + 4(k + 1)$. This is divisible by 3 if and only if $k + 1$ is divisible by 3. That is, if $k = 2$, $k = 5$, or $k = 8$. We see that a shuffle number must have 2 eights and 8 fours, or 5 eights and 5 fours, or 8 eights and 2 fours. Note that each of those numbers satisfy (1) and (3). It remains to examine which of these numbers are divisible by 11 (property (2)).

For this we use the 11-criterion: a number is divisible by 11 if the *alternating sum* of the digits is divisible by 11. By alternating sum we mean that instead of adding them, we alternately add and subtract. Since in our case all digits are equal to 4 or to 8, the alternating sum must even be divisible by $4 \times 11 = 44$. Since the alternating sum cannot be greater than $8+8+8+8+8-4-4-4-4-4 = 20$ (and cannot be smaller than -20), it must be equal to 0. In other words: the sum of the five digits in the odd positions must equal the sum of the five digits in the even positions. This means that the number of digits 8 in the odd positions must be equal to the number of digits 8 in the even positions. We examine the three cases that we found before:

- Suppose exactly 2 digits are eights. The only requirement is now that there is exactly 1 eight in the odd positions and exactly 1 eight in the even positions. There are $5 \times 5 = 25$ ways to do this.
- Suppose exactly 5 digits are eights. Since we cannot divide the eights into two equal groups, there are no solutions in this case.
- Suppose exactly 8 digits are eights. The only requirement is now that we have 4 eights in the odd positions and 4 eights in the even positions. In other words: 1 odd position must be a four and 1 even position must be a four. Again we find $5 \times 5 = 25$ possibilities.

We conclude that the total number of 10-digit shuffle numbers is $25 + 25 = 50$.

2. Version for klas 5 & klas 4 and below

We first consider the case that 1 is red. Then all numbers from 1 to 25 are red. Indeed, suppose that some number k is blue. Then 1 and k have different colours, hence by the third rule the number $1 \cdot k = k$ must be red. But this contradicts the assumption that it was blue. Observe that colouring all numbers red indeed gives a correct colouring.

Now consider the case that 1 is blue. Since 5 is not blue, it follows from the second rule that 1 and 4 have the same colour. Hence, 4 is blue. Similarly, 2 and 3 must have the same colour. In fact, both must be blue since if 2 were red, then $3 = 1 + 2$ would have to be blue because of the second rule.

We now know that 1, 2, 3, and 4 are blue. By the second rule, we see that also $6 = 5 + 1$, $7 = 5 + 2$, $8 = 5 + 3$, and $9 = 5 + 4$ are blue. Applying the second rule again, we find in turn that $11, 12, 13, 14$ are blue, that $16, 17, 18, 19$ are blue, and that $21, 22, 23, 24$ are blue.

Because of the third rule, $10 = 2 \cdot 5$, $15 = 3 \cdot 5$, and $20 = 4 \cdot 5$ are red. Only the colour of 25 is not yet determined by the rules. We find two possible colourings:

(1) Only $5, 10, 15, 20, 25$ (the numbers divisible by 5) are red.

(2) Only $5, 10, 15, 20$ are red.

We check that colouring (1) is indeed correct. The second rule is satisfied, because the sum of a number divisible by 5 and a number not divisible by 5 is itself not divisible by 5. The third rule is satisfied since the product of a number divisible by 5 and a number not divisible by 5 is itself divisible by 5.

Colouring (2) differs from colouring (1) only in the colour of 25. To see that colouring (2) is correct, we only need to check the rules in the situation that one of the numbers involved is 25. We check that 25 is not the product of a blue and a red number (that is correct). We must also check that there is no red number k such that $25 + k$ is red or $25k$ is blue. That is automatically true since the numbers greater than 25 are not coloured.

We conclude that there are 3 correct colourings altogether: the colouring in which all numbers are red, colouring (1), and colouring (2).

2. Version for klas 6

We first consider the case that 1 is red. Then all numbers from 1 to 15 are red. Indeed, suppose that some number k is blue. Then 1 and k have different colours, hence by the third rule the number $1 \cdot k = k$ must be red. But this contradicts the assumption that it was blue. Observe that colouring all numbers red indeed gives a correct colouring.

Now we consider the case that 1 is blue. Observe that when two numbers sum to 15, those numbers must have the same colour by the second rule. We get the following pairs of numbers of the same colour: 1 and 14, 2 and 13, 3 and 12, 4 and 11, 5 and 10, 6 and 9, 7 and 8.

The number 2 is blue. Indeed, suppose that 2 is red. From the second rule, we derive that $3 = 1 + 2$ is blue. Repeatedly applying the same rule we find that $5 = 3 + 2$ is blue, that $7 = 5 + 2$ is blue, and finally that 15 is blue. Since 15 is in fact not blue, 2 cannot be red.

The number 7 is blue. Indeed, suppose that 7 is red. It then follows from the second rule that $8 = 1 + 7$ is blue. However, 7 and 8 must have the same colour, so this cannot be the case.

The number 4 is blue. Indeed, suppose that 4 is red. It then follows from the second rule that $11 = 4 + 7$ is blue. But 4 and 11 have the same colour, so this cannot be the case.

Recall, that 3 and 12 have the same colour, and so do 6 and 9. In fact, all four numbers must have the same colour. Indeed, otherwise $9 = 3 + 6$ would be blue by the second rule and also $12 = 3 + 9$ would be blue by the second rule.

So far, we know that 15 is red, that $1, 2, 4, 7, 8, 11, 13, 14$ are blue, that $3, 6, 9, 12$ have the same colour, and that $5, 10$ have the same colour. The numbers 3 and 5 cannot both be red, since if 3 is red, $5 = 2 + 3$ is blue by the second rule. The three remaining colour combinations for the numbers 3 and 5 result in the following three colourings:

(1) Only 15 is red.

(2) Only $5, 10, 15$ (the numbers divisible by 5) are red.

(3) Only $3, 6, 9, 12, 15$ (the numbers divisible by 3) are red.

It is easy to check that all three colourings are indeed correct. We write this out for the third colouring, the other two can be checked in a similar way. That the sum of a red and a blue number is always blue follows from the fact that the sum of a number divisible by 3 and a number not divisible by 3 is itself not divisible by 3. That the product of a blue and a red number is always red follows from the fact that the product of a number divisible by 3 and a number not divisible by 3 is itself divisible by 3.

We conclude that there are 4 correct colourings in total: the colouring in which all numbers are red, and colourings (1), (2), and (3).

3. If we subtract the second equation from the first, we obtain

$$x^2 - z^2 = -x + z - (x - z),$$

which can be rewritten as

$$(x - z)(x + z) = -2(x - z),$$

and then simplified further to get

$$(x - z)(x + z + 2) = 0.$$

Since x and z must be different, $x - z$ is nonzero and we can conclude that $x + z + 2 = 0$, hence $z = -x - 2$. If we subtract the third equation from the second, we obtain

$$y^2 - x^2 = x + 3y - (2x + 2y),$$

which can be rewritten as

$$(y - x)(y + x) = 1(y - x),$$

and then simplified further to get

$$(y - x)(y + x - 1) = 0.$$

Since x and y must be different, $y - x$ is nonzero and we can conclude that $y + x - 1 = 0$, hence $y = 1 - x$.

If we now substitute $y = 1 - x$ and $z = -x - 2$ in the first equation, we get

$$x^2 + (1 - x)^2 = -x + 3(1 - x) + (-x - 2),$$

which we can rewrite as

$$2x^2 - 2x + 1 = -5x + 1.$$

This is a quadratic equation $2x^2 + 3x = 0$, whose roots are $x = 0$ and $x = -\frac{3}{2}$. In both cases we can deduce the values of y and z from the formulas $y = 1 - x$ and $z = -x - 2$. This gives the solutions $(x, y, z) = (-\frac{3}{2}, \frac{5}{2}, -\frac{1}{2})$ and $(x, y, z) = (0, 1, -2)$.

For these solutions we know that they satisfy the first equation. Since the difference between the first and the second equation gives zero on both sides (since we chose $z = -x - 2$), the second equation is satisfied as well. Because of the choice $y = 1 - x$, also the third equation is satisfied. We conclude that both solutions indeed satisfy all three equations.

4. Version for klas 4 & below

(a) Because BE is the bisector of $\angle ABC$, it follows that $\angle ABE = \angle CBF$. It was given that $\angle FCB = \angle EAB$. Since the angles in a triangle sum to $180°$, both in triangle ABE and in triangle CBF, we must also have $\angle BFC = \angle BEA$. Since $\angle BFC$ and $\angle AFE$ are opposite angles, we see that $\angle AFE = \angle FEA$, and hence that triangle AEF is isosceles.

(b) We will show that $\triangle ABF$ and $\triangle EGA$ are congruent.

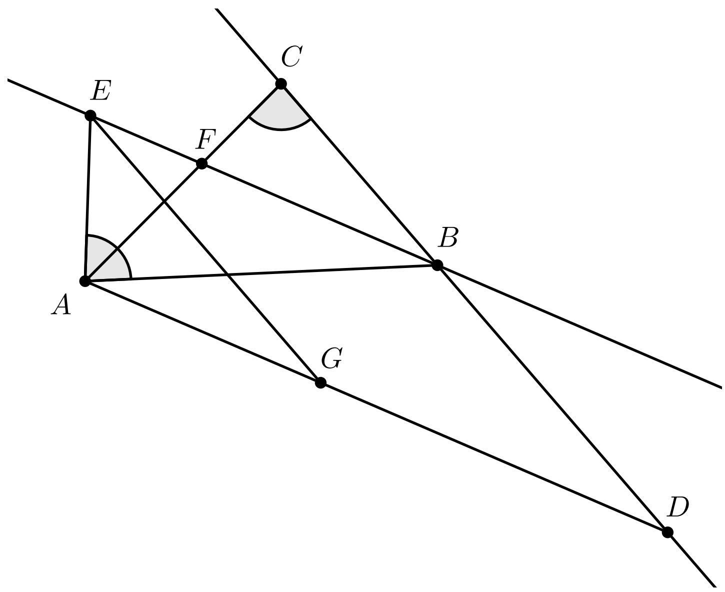
We see that $\angle EGA = \angle CDA$, since EG and BC are parallel. In the isosceles triangle ABD, we see that $\angle CDA = \frac{1}{2}(180° - \angle ABD)$, which is equal to $\frac{1}{2}\angle ABC$ (because of the straight angle). This in turn, is equal to $\angle ABF$, because BF is the bisector of $\angle ABC$. Altogether, we find that $\angle EGA = \angle ABF$.

Triangle ABD is isosceles, so $\angle BAG = \angle CDA$. As we just observed, this last angle is equal to $\angle ABF$. We have also seen that $\angle EGA = \angle ABF$. Because the three angles in triangle GEA sum to $180°$, we get

$$\angle GEA = 180° - \angle EGA - \angle EAB - \angle BAG = 180° - \angle ABF - \angle FCB - \angle ABF.$$

Since $2 \cdot \angle ABF = \angle ABC$ (since BF is the bisector) we thus find that $\angle GEA = 180° - \angle FCB - \angle ABC = \angle BAC$ (because the angles in triangle ABC sum to $180°$), hence $\angle GEA = \angle BAF$.

Using the fact that $|AE| = |AF|$, which follows from the fact that $\triangle AEF$ is isosceles, we can now conclude that $\triangle ABF$ and $\triangle EGA$ are congruent. This immediately implies $|AG| = |BF|$ as required.



4. Version for klas 5 & klas 6

See the solution of **version for klas 4 & below**. There we first prove that triangle AEF is isosceles as an intermediate step (part (a)).

5. The smallest number of questions that suffices is 32. First we will show a strategy to locate the prize in only 32 questions.

Start by asking the quizmaster if the prize is behind the left-hand door. Repeat this until you are sure whether the prize is there or not. You can only be 100% sure when you have received the same answer 11 times, because the quizmaster can lie a maximum of 10 times. After doing this, suppose that the quizmaster has lied n times. This means that you have thus far asked $11 + n$ questions and the quizmaster is entitled to $10 - n$ more lies.

Now ask the quizmaster whether the prize is behind the right-hand door, and keep asking this until you are 100% sure of the true answer. Since the quizmaster can lie only $10 - n$ more times, you need at most $2(10 - n) + 1 = 20 - 2n + 1$ questions for this. In total you have now asked $32 - n$ questions. We conclude that with this strategy, 32 questions always suffice.

Having only 31 questions, it is not always possible to locate the prize. We will show how the quizmaster can make sure of that. At the beginning, as long as you have asked about both doors at most 10 times, he always answers 'no'. To keep it simple, we assume that the left-hand door is the first door that you query for the eleventh time (the other case is completely analogous). We consider the situation that the prize is not behind the left-hand door (which may happen). In this case, we show that the quizmaster can make sure that after 31 questions, you cannot tell whether the prize is behind the middle door or behind the right-hand door.

So far, the quizmaster has been speaking the truth about the left-hand door, and he will continue to do so: when asked about the left-hand door, he will always answer 'no'. For the right-hand door, the quizmaster will keep saying 'no' up to and including the tenth time he is asked about this door. The next ten times he is asked about this door, he will answer 'yes'. Since you ask at most 20 questions about the right-hand door (since you already queried the left-hand door at least 11 times), the quizmaster needs to lie at most 10 times. Whether the prize is behind the middle door or behind the right-hand door, in both cases the quizmaster gives the same 31 answers. Therefore, you cannot determine the door behind which the prize is located.