Final round
Dutch Mathematical Olympiad

Friday 15 September 2017

Solutions

1. Version for klas 5 & klas 4 and below

An integer for which the digits (from left to right) are $c_1, c_2, \ldots, c_k$ will be denoted by $\overline{c_1c_2\ldots c_k}$.

(a) Let $n = \overline{c_1c_2\ldots c_k}$ be an even-steven integer greater than 9 (hence $k \geqslant 2$). We will prove that n is indeed the sum of two oddball integers.

Let $p = \overline{c_2\ldots c_k}$ and $q = \overline{c_1 0\ldots 0}$ (with $k-1$ digits equal to 0). As n is even-steven, we have $c_2 \geqslant c_1 \geqslant 1$, hence p and q do not start with the digit 0. Moreover, it is clear that $n = p + q$. Hence, it suffices to prove that p and q are oddball.

The integer q is oddball, because all digits except for the first are equal to 0. The digits of p are the same digits as the digits of n, except for the first digit c_1. However, the digits that were at an even position in n are at an odd position in p and vice versa. Therefore, also p is oddball.

(b) The integer $n = 109$ is oddball, but it is not the sum of two even-steven integers. We will prove this by contradiction. Suppose that $n = p + q$ for some even-steven integers p and q. We will show that this leads to a contradiction.

First observe that the integers 100 to 108 are not even-steven. Hence, both p and q must be smaller than 100, and hence both are also greater than 9. In other words, p and q have exactly two digits. Suppose $p = \overline{ab}$ and $q = \overline{cd}$. The equation $p + q = 109$ now yields $b + d = 9$ (because $b + d < 19$) and $a + c = 10$. Hence, $b + d < a + c$, which implies that either $b < a$ or $d < c$ (or both). In the first case p is not even-steven and in the second case q is not even-steven. This contradicts the assumption that p and q are even-steven.

We conclude that the oddball integer 109 cannot be written as the sum of two even-steven integers.

1. Version for klas 6

An integer for which the digits (from left to right) are $c_1, c_2, \ldots, c_k$ will be denoted by $\overline{c_1c_2\ldots c_k}$.

(a) Let $n = \overline{c_1c_2\ldots c_k}$ be an oddball integer greater than 9 (hence $k \geqslant 2$). We will show that n is indeed the sum of two oddball integers.

If $c_2 \geqslant 1$, then we can write n as the sum of the following two oddball integers: $\overline{10\ldots0}$ ($k-2$ zeros) and $\overline{c_1(c_2{-}1)c_3\ldots c_k}$.

If $c_2 = 0$ and $c_1 \geqslant 2$, then we can write n as the sum of the following two oddball integers: $\overline{10\ldots0}$ ($k-1$ zeros) and $\overline{(c_1{-}1)c_2c_3\ldots c_k}$.

If $n = \overline{10\ldots0}$, then we can write n as the sum of the following two oddball integers: $\overline{1}$ and $\overline{9\ldots9}$ ($k-1$ nines).

The last case is the case in which $c_1 = 1$, $c_2 = 0$ and not all digits $c_3, \ldots, c_k$ are equal to 0. Let c_t be a digit unequal to 0, with $t \geqslant 3$ as small as possible. Hence, $n = \overline{10\ldots0c_t\ldots c_k}$ with $c_t \geqslant 1$.

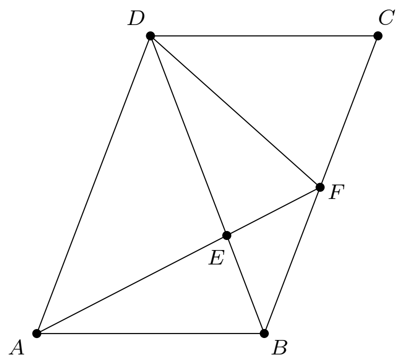
Because n is oddball and $c_{t-1} = 0 < 1 \leqslant c_t$, we find that t must be *odd*. We can now write n as the sum of the integers $\overline{10\ldots0}$ ($k-1$ zeros) and $m = \overline{c_tc_{t+1}\ldots c_k}$. Because t is odd, the digits at the odd positions of m are also at odd positions of n. Therefore, these digits are greater than or equal to their neighbouring digits (because n is oddball), which yields that m is oddball.

(b) The integer $n = 109$ is oddball, but it is not the sum of two even-steven integers. We will prove this by contradiction. Suppose that $n = p + q$ for some even-steven integers p and q. We will show that this leads to a contradiction.

First observe that the integers 100 to 108 are not even-steven. Hence, both p and q must be smaller than 100, and hence both are also greater than 9. In other words, p and q have exactly two digits. Suppose $p = \overline{ab}$ and $q = \overline{cd}$. The equation $p + q = 109$ now yields $b + d = 9$ (because $b + d < 19$) and $a + c = 10$. Hence, $b + d < a + c$, which implies that either $b < a$ or $d < c$ (or both). In the first case p is not even-steven and in the second case q is not even-steven. This contradicts the assumption that p and q are even-steven.

We conclude that the oddball integer 109 cannot be written as the sum of two even-steven integers.

2. Version for klas 4 & below



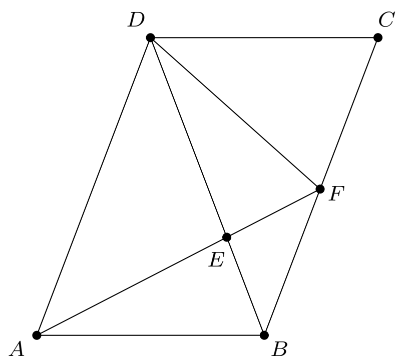
Since AED is an isosceles triangle, angles $\angle EDA$ and $\angle DAE$ are equal. In turn, these angles are equal to angles $\angle EBF$ and $\angle BFE$ (alternate interior angles). This implies that triangle BFE is isosceles as well, with $|BE| = |EF|$.

Comparing triangles ABE and DFE, we see that $|AE| = |DE|$ and $|BE| = |FE|$. Since $\angle BEA$ and $\angle FED$ are a pair of opposite angles, they have the same size. It follows that triangles ABE and DFE are congruent (SAS).

From the fact that ABE and DFE are congruent it follows that $|DF| = |AB|$. Since $ABCD$ is a parallelogram, we also have $|AB| = |CD|$. It follows that triangle CDF is isosceles as well (with apex D).

Triangle ADB is also isosceles. Since $\angle FCD = \angle BAD$ (because $ABCD$ is a parallelogram), we deduce that triangles CDF and ADB are similar. In particular, $\angle CDF = \angle ADB$ holds.

2. Version for klas 5 & klas 6



Since AED is an isosceles triangle, angles $\angle EDA$ and $\angle DAE$ are equal. In turn, these angles are equal to angles $\angle EBF$ and $\angle BFE$ (alternate interior angles). This implies that triangle BFE is isosceles as well, with $|BE| = |EF|$.

Comparing triangles ABE and DFE, we see that $|AE| = |DE|$ and $|BE| = |FE|$. Since $\angle BEA$ and $\angle FED$ are a pair of opposite angles, they have the same size. It follows that triangles ABE and DFE are congruent (SAS).

From the fact that ABE and DFE are congruent it follows that $|DF| = |AB|$. Since $ABCD$ is a parallelogram, we also have $|AB| = |CD|$. It follows that triangle CDF is isosceles as well (with apex D).

On the one hand, this implies that $\angle FCD = \angle DFC$. On the other hand, we know that triangle DBC is isosceles (since $|BD| = |AD| = |BC|$), which implies that $\angle FCD = \angle CDB = 2 \cdot \angle CDF$ since DF is the angle bisector of $\angle CDB$.

Altogether, we have $\angle DFC = \angle FCD = 2 \cdot \angle CDF$. Since the angles in any triangle sum to 180 degrees, we also know that

$$180^\circ = \angle DFC + \angle FCD + \angle CDF = 5 \cdot \angle CDF.$$

From this, it follows that $\angle CDF = \frac{1}{5} \cdot 180^\circ = 36^\circ$, and hence $\angle FCD = 2 \cdot \angle CDF = 72^\circ$. Since triangle DBC is isosceles, also $\angle CDB = 72^\circ$ holds. Using alternating interior angles, we now find that $\angle ABD = \angle CDB = 72^\circ$.

3. Let the scores of the six teams be s, $s+2$, $s+4$, $s+6$, $s+8$, and $s+10$. Let T be the total number of awarded points, so that $T = 6s+30$. It follows that the total number of points is a multiple of six.

The number of games played equals $\frac{6\cdot 5}{2} = 15$. Let g be the number of games that ended in a draw. A game that ends in a draw results in $1+1=2$ awarded points and every other game results in $3+0=3$ awarded points. Therefore, the total number of awarded points equals $T = g\cdot 2 + (15-g)\cdot 3 = 45-g$.

From $T = 45-g$ it follows that $30 \leqslant T \leqslant 45$ because the number of draws satisfies $0 \leqslant g \leqslant 15$. Since T is a multiple of six, this leaves the following possibilities: $T = 30$, $T = 36$, and $T = 42$.

If $T = 30$, we have $g = 45-30 = 15$. But then all games must have ended in a draw and all teams must have the same score. Hence, the case $T = 30$ is ruled out.

If $T = 36$, then $g = 45-36 = 9$ and $s = \frac{T-30}{6} = 1$. The six scores are therefore $1, 3, 5, 7, 9, 11$. The team that scored 1 point must have lost 4 games (and played one draw). The team that scored 3 points must have lost at least 2 games (at most 3 games were not lost). The team that scored 11 points must have won at least 3 games (otherwise the score is at most $3+3+1+1+1 = 9$), so apart from the teams with scores 1 and 3 at least one other team has lost a game. In total, at least $4+2+1 = 7$ games ended in a loss for some team, contradicting the fact that $15-9 = 6$ games did not end in a draw. This rules out the case $T = 36$.

Finally, we consider that case $T = 42$. The six scores are $2, 4, 6, 8, 10, 12$ and we have $g = 3$. Since the total number of points obtained from won games is a multiple of three, the six teams must have received at least $2, 1, 0, 2, 1, 0$ points from draws, respectively. In total, exactly $2\cdot 3 = 6$ points are awarded in games that ended in a draw. Hence, since $2+1+0+2+1+0 = 6$, the six teams have received exactly the mentioned numbers of points from draws. In particular, the team ending in the fourth place (with 6 points), was involved in 0 draws and must have won exactly two games.

4. Version for klas 5 & klas 4 and below

(a) Let r be the remainder upon dividing n by a. We will first prove that $r < \frac{n}{2}$. If $2a \leqslant n$, this follows from the fact that $r < a$. If $2a > n$, we have $r = n-a$ (since we already knew that $a < n$), which implies that $r = n-a < n - \frac{n}{2} = \frac{n}{2}$.

For the same reasons, the remainder upon dividing n by b is smaller than $\frac{n}{2}$.

It follows that the two remainders obtained by dividing n by a and b add up to a number smaller than n.

(b) Let r, s, and t be the remainders upon dividing n by 99, 132, and 229, respectively. The number $n-t$ is a multiple of 229 and nonzero since $n > 229 > t$. We know that $r+s+t = n$, and hence $n-t = r+s$. We can conclude that $r+s$ is a positive multiple of 229. Since $99+132 < 2\cdot 229$, we have $r+s < 2\cdot 229$, which implies that we must have $r+s = 229$.

(c) Since $r \leqslant 98$ and $s \leqslant 131$, the fact that $r+s = 229$ implies that $r = 98$ and $s = 131$. Therefore, the number $n+1$ is divisible by both 99 and 132, and hence by their least common multiple $\text{lcm}(99, 132) = \text{lcm}(9\cdot 11, 3\cdot 4\cdot 11) = 4\cdot 9\cdot 11 = 396$. Also, from $n = 229+t$ and $t < 229$ we deduce that $n+1 \leqslant 458$. It follows that the only possibility is $n+1 = 396$, hence $n = 395$.

When $n = 395$ the three remainders are $r = 98$, $s = 131$, and $t = 166$, and indeed satisfy the equation $n = r+s+t$.

4. Version for klas 6

(a) Let r be the remainder upon dividing n by a. We will first prove that $r < \frac{n}{2}$. If $2a \leqslant n$, this follows from the fact that $r < a$. If $2a > n$, we have $r = n - a$ (since we already knew that $a < n$), which implies that $r = n - a < n - \frac{n}{2} = \frac{n}{2}$.

For the same reasons, the remainder upon dividing n by b is smaller than $\frac{n}{2}$.

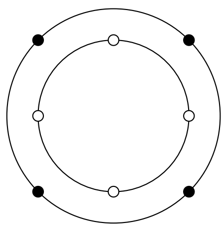
It follows that the two remainders obtained by dividing n by a and b add up to a number smaller than n.

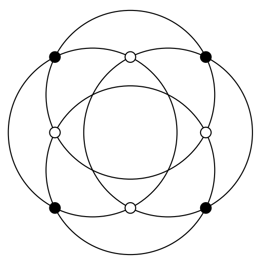
(b) Let r, s, and t be the remainders upon dividing n by 99, 132, and 229, respectively. The number $n-t$ is a multiple of 229 and nonzero since $n > 229 > t$. We know that $r+s+t = n$, and hence $n - t = r + s$. We can conclude that $r + s$ is a positive multiple of 229. Since $99 + 132 < 2 \cdot 229$, we have $r + s < 2 \cdot 229$, which implies that we must have $r + s = 229$.

Since $r \leqslant 98$ and $s \leqslant 131$, the fact that $r + s = 229$ implies that $r = 98$ and $s = 131$. Therefore, the number $n + 1$ is divisible by both 99 and 132, and hence by their least common multiple $\text{lcm}(99, 132) = \text{lcm}(9\cdot 11, 3\cdot 4\cdot 11) = 4\cdot 9\cdot 11 = 396$. Also, from $n = 229+t$ and $t < 229$ we deduce that $n + 1 \leqslant 458$. It follows that the only possibility is $n + 1 = 396$, hence $n = 395$.

When $n = 395$ the three remainders are $r = 98$, $s = 131$, and $t = 166$, and indeed satisfy the equation $n = r + s + t$.

5. Four of the eight points are coloured black and the other four points are coloured white in the way indicated in the figure on the left. The circle through the four black points is denoted C_1 and the circle through the four white points is denoted C_2. If two points lie on a circle C, we say that C *covers* that pair of points.





Circle C_1 covers all pairs of black points and circle C_2 covers all pairs of white points. It is easy to check that each of the $4 \cdot 4 = 16$ pairs consisting of a white point and a black point is covered by one of the four circles in the figure on the right. It follows that the six circles form a solution.

We will now prove that five or fewer circles do not suffice. First observe that any circle passing through more than two black points must be equal to C_1 and that any circle passing through more than two white points must be equal to C_2. Indeed, a circle is already determined by three points.

A circle passing through 2 or fewer black points covers at most one of the $\frac{4\cdot 3}{2} = 6$ pairs of black points. A solution consisting of only five circles must therefore contain circle C_1 (since otherwise at most 5 pairs of black points are covered). In the same way we see that such a solution must contain circle C_2.

Each of the remaining three circles in the (hypothetical) solution contains at most 2 black points and at most 2 white points. Such a circle covers at most $2 \cdot 2 = 4$ pairs consisting of a white and a black point. In total, the five circles can therefore cover at most $0 + 0 + 3 \cdot 4 = 12$ such pairs, while there are 16 to be covered. The five circles can therefore not form a correct solution after all. We conclude that the smallest number of circles in a solution is 6.