Final Round
Dutch Mathematical Olympiad

Friday 18 September 2015

Solutions

1. (a) It is possible to make 2015 groups. For example, take the 2015 groups $\{-4, -3, -2, -1, i\}$, where i runs from 1 to 2015. Each group consists of five distinct numbers, as required, and any two groups have exactly four numbers in common: -4, -3, -2, and -1.

(b) Using six available numbers, there are only six possible groups of five numbers (each obtained by leaving out one of the six numbers). Those six groups do satisfy the requirement that any two of them have exactly four numbers in common. We conclude that six is the greatest number of groups we can make in this case.

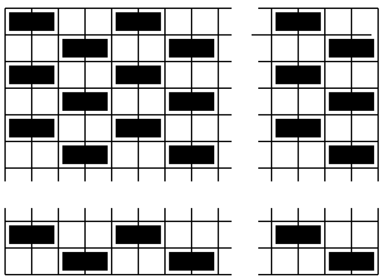
(c) A way to make three groups is to take $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 6\}$, and $\{1, 2, 3, 4, 7\}$.

More than three groups is not possible. Indeed, suppose we have four or more groups. The first two groups are $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, f\}$, where a, b, c, d, e, and f are distinct numbers. Then there must be a third group C containing a seventh number g. The remaining four numbers in C must be in both A and B, hence $C = \{a, b, c, d, g\}$.

Now consider a hypothetical fourth group D. This group cannot contain the number g since otherwise, using a similar reasoning as for C, we would have $D = \{a, b, c, d, g\}$. Because D does not contain the number g, it must contain the remaining four numbers a, b, c, and d from C. Comparison with groups A and B then shows that D can contain neither e nor f. It follows that besides a, b, c, and d, D cannot contain a fifth number, contradicting the requirements.

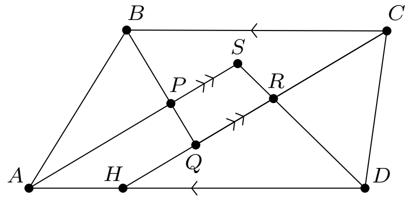
We conclude that the greatest number of groups we can make is three.

2. A mamimum of 250,000 dominoes can fit on the board. We first show to place this number of dominoes on the board. In each row we put 250 dominoes with two empty squares in between consecutive dominoes. In the odd numbered rows we start with a domino and end with two empty squares (since the number of squares in a row is a multiple of four). In the even numbered rows we start with two empty squares and end with a domino. Thus, we place a total of $1000 \cdot 250 = 250{,}000$ dominoes, see the figure. Clearly, no two dominoes in the same row are adjacent, and dominoes in adjacent rows touch in a vertex, at most.



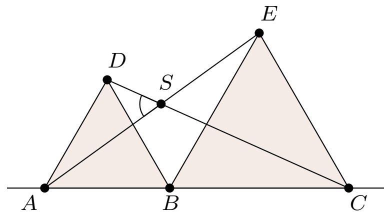
To complete the proof, we need to show that it is not possible to place more than 250,000 dominoes on the board. Partition the board into 500×500 patches consisting of 2×2 squares each. Of each patch, at most two out of the four squares can be covered by dominoes since otherwise two dominoes would be adjacent. Hence, no more than $2 \cdot 500 \cdot 500 = 500{,}000$ squares can be covered by dominoes. This shows that no more than 250,000 dominoes can fit on the board.

3. **[Versie klas 5 & klas 4 en lager]** The intersection of CQ and AD is called H. We have $\frac{1}{2}\angle BAD = \angle SAD = \angle CHD$ (corresponding angles). Also, we have $\angle CHD = \angle HCB = \frac{1}{2}\angle DCB$ (alternate interior angles). It follows that the two angles $\angle BAD$ and $\angle DCB$ of quadrilateral $ABCD$ are equal. This implies that $\angle BAD + \angle ADC = \angle DCB + \angle ADC = 180°$, because AD and BC are parallel. From $\angle BAD + \angle ADC = 180°$ it follows that AB and CD are parallel. Hence, $ABCD$ is a parallelogram. We conclude that $|AB| = |CD|$.



3. **[Versie klas 6]** Observe that $\angle ABE = 180° - \angle EBC = 120°$ and $\angle DBC = 180° - \angle ABD = 120°$. Furthermore, $|AB| = |DB|$ and $|BE| = |BC|$. It follows that triangles ABE and DBC are congruent (SAS). In particular, $\angle EAB = \angle CDB$.

Observe that $\angle ASD = 180° - \angle SDA - \angle DAS = 180° - (60° + \angle CDB) - \angle DAE$. Substituting $\angle CDB = \angle EAB$ shows that $\angle ASD = 120° - \angle EAB - \angle DAE = 120° - 60° = 60°$.



4. We start by observing that in the equation $7pq^2 + p = q^3 + 43p^3 + 1$ the numbers p and q cannot both be odd. Otherwise, $7pq^2 + p$ would be even, while $q^3 + 43p^3 + 1$ would be odd. Since 2 is the only even prime number, we conclude that $p = 2$ or $q = 2$.

In the case $p = 2$, we obtain the equation $14q^2 + 2 = q^3 + 344 + 1$, which can be rewritten as $q^3 - 14q^2 = -343$. This shows that q must be a divisor of $343 = 7^3$, hence $q = 7$. Substitution confirms that $(p, q) = (2, 7)$ is indeed a solution since $14q^2 + 2 = 2 \cdot 7 \cdot 7^2 + 2$ and $q^3 + 344 + 1 = 7^3 + (7^3 + 1) + 1 = 2 \cdot 7^3 + 2$ are equal.

Next, we consider the case that $q = 2$ and p is odd. This results in the equation $28p + p = 8 + 43p^3 + 1$. Since p is odd, we see that $28p + p$ is odd, while $8 + 43p^3 + 1$ is even. Hence, no solutions exist in this case.

We conclude that $(p, q) = (2, 7)$ is the only solution.

5. The system of inequalities is symmetric in the variables a, b, and c: if we exchange two of these variables, the system remains unchanged (up to rewiting it). For example, if we exchange variables a and b, we obtain

$$|b - a| \geqslant |c|, \quad |a - c| \geqslant |b|, \quad |c - b| \geqslant |a|.$$

Since $|b - a| = |a - b|$, $|a - c| = |c - a|$, and $|c - b| = |b - c|$, this can be rewritten as

$$|a - b| \geqslant |c|, \quad |c - a| \geqslant |b|, \quad |b - c| \geqslant |a|,$$

obtaining the original system of inequalities. Due to this symmetry, we may assume without loss of generality that $a \geqslant b \geqslant c$.

First observe that $c \leqslant 0$. Indeed, if $a \geqslant b \geqslant c > 0$ would hold, then $|b - c| \geqslant |a|$ would imply that $b - c \geqslant a$. Rewriting gives $b \geqslant a + c > a$, which contradicts the fact that $b \leqslant a$.

Next, consider the following series of inequalities.

$$|a| + |c| = |a| - c \geqslant a - c = (a - b) + (b - c) = |a - b| + |b - c| \geqslant |c| + |a|.$$

Since the first and the last term are equal, we can conclude that all of the above inequalities must be equalities. In particular, $a - b = |a - b| = |c|$. Since $c \leqslant 0$, this implies that $a + c = b$. This shows that one of the three numbers equals the sum of the other two.

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