

# Second round

## Dutch Mathematical Olympiad



Friday, March 14, 2025 ( $\pi$ -day)

Solutions

B-problems

- B1.**  $2025 : 1024$  To see what the new ratio of the sides of the rectangle is after cutting once, we can divide the largest number in the ratio by two, or just multiply the smallest number by two. So we see that after the first 11 cuts, the ratios are successively

$$2025 : 2, \quad 2025 : 4, \quad 2025 : 8, \quad \dots, \quad 2025 : 1024, \quad 2025 : 2048,$$

since  $2^{11} = 2048$ . Now the second side has just become the longest side, and after cutting one more time, the ratio again becomes  $2025 : 1024$ . We see that the ratio becomes alternately  $2025 : 2048$  and  $2025 : 1024$ . Since we cut an even number of times in total, the ratio after 100 cuts is therefore the same as the ratio after 12 cuts:  $2025 : 1024$ .

- B2.**  $61$  We can start with digits 1 to 9. After that, we have to choose three times whether the next digit is one more or one less; there are eight ways to do that. So in total, there are at most 72 rolling numbers of 4 digits. We do have to be careful that a digit cannot become smaller than 0 or larger than 9.

If we start with a 1, the third digit cannot be two smaller than the first. This rules out 2 of the possibilities, leaving 1010, 1012, 1210, 1212, 1232 and 1234. Similarly, there are only 6 possibilities for the starting digit 8.

For the initial digit 2, one possibility is ruled out because the last digit cannot be three smaller than the first. Similarly, for the initial digit 7, there are only 7 possibilities.

After the starting digit 9, an 8 should immediately follow. This can be followed by 76, 78 or 98. So here are only 3 possibilities.

For all other initial digits, there are as many as eight possibilities. Below is a table showing the number of possibilities for each initial digit.

first digit	1	2	3	4	5	6	7	8	9
number of rolling numbers	6	7	8	8	8	8	7	6	3

So, in total, there are  $6 + 7 + 8 + 8 + 8 + 8 + 7 + 6 + 3 = 61$  rolling 4-digit numbers.

- B3.**  $41$  We go through the possibilities systematically. Suppose Sam colours the number 1 blue. Then 2 and 3 must be coloured red, but then there are no restrictions on the colours from 4 to 7. This can be done in  $2^4 = 16$  ways.

On the contrary, suppose Sam colours the number 1 red. Then 2 and 3 may be either red or blue. We first look at the possibilities for the numbers 2, 4 and 5. If 2 is blue, then 4 and 5 must be red, and if 2 is red, then there are four ways to colour 4 and 5. So in total, there are 5 ways to colour 2, 4 and 5 in this case. Similarly, we see that completely independently, there are 5 ways to colour the numbers 3, 6 and 7. So in total there are  $5 \cdot 5 = 25$  ways to colour the numbers when 1 is coloured red.

So all in all, we find  $16 + 25 = 41$  possible colourings.

- B4.** 300 Say  $A$  pieces fit in a row and  $B$  pieces fit in a column. Then there are  $A - 2$  pieces on the top edge,  $A - 2$  pieces on the bottom edge,  $B - 2$  pieces on the left edge and  $B - 2$  pieces on the right edge. In total, there are  $A \cdot B$  pieces. So we can write down the observation in the problem as

$$4 + 2(A - 2) + 2(B - 2) = \frac{22}{100}AB.$$

We can rewrite this as

$$11AB = 100A + 100B - 200 = 100(A + B - 2).$$

Since the right-hand side is a multiple of 100, the number  $11AB$  must also be a multiple of 100. The prime factors 2 and 5 of 100 do not occur in 11, so  $AB$  must be a multiple of  $100 = 2 \cdot 2 \cdot 5 \cdot 5$ . We consider the possible distributions of the factors 2, 2, 5 and 5 over  $A$  and  $B$ . To simplify calculations, we rewrite  $11AB = 100(A + B - 2)$  to

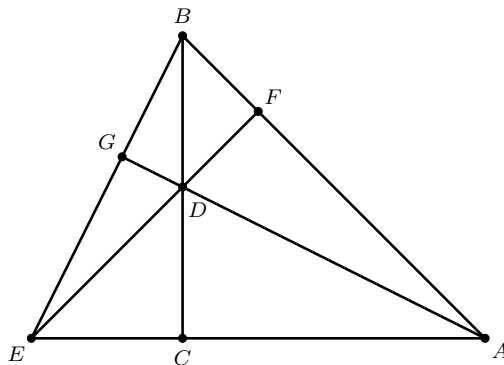
$$B = \frac{100A - 200}{11A - 100}.$$

If both factors 5 are in  $A$ , then  $A$  is divisible by 25 and therefore equal to 25 or 50. For  $A = 25$  we get  $B = \frac{92}{7}$  and for  $A = 50$  we find  $B = \frac{32}{3}$ , both not integers. So the two factors 5 are not both in  $A$ . By the same argument with  $A$  and  $B$  interchanged, we see that the two factors 5 are not both in  $B$ . So both  $A$  and  $B$  are divisible by 5.

Further, the factors 2 must also be distributed. Again by interchanging  $A$  and  $B$  if necessary, we may assume that  $A$  contains a factor 2, hence  $A$  is a multiple of 10. We consider the possibilities  $A = 10, 20, 30, 40$ . For  $A = 10$ , the equation gives that  $B$  should be equal to 80, but it can't be. For  $A = 20$  we get  $B = 15$ . For  $A = 30$  we get  $B = \frac{280}{23}$ , which is not integer. For  $A = 40$  we find  $B = \frac{190}{17}$  and that too is not an integer.

So the only possibility is  $A = 20$  and  $B = 15$  (or vice versa) and so in total there are  $20 \cdot 15 = 300$  pieces.

- B5.**  $6\sqrt{10}$  Let  $F$  be the intersection of  $ED$  and  $AB$ . Triangle  $ABC$  is a 'set square' with angles of 45, 45 and 90 degrees. Because of the right angles at  $F$  and  $C$  and the opposite angles at  $D$ , we see that there are two more of these 'set squares': triangle  $BDF$  and triangle  $EDC$ . Of the latter, the length of the short side  $CD$  is exactly half the length of the short side  $AC$  of triangle  $ABC$ . It follows that  $|AE| = \frac{3}{2}|AC|$ . Triangle  $ABE$  therefore has  $\frac{3}{2}$  times as long a base and the same height  $BC$  as triangle  $ABC$ . So the area of triangle  $ABE$  is 150.



Moreover, triangle  $ABC$  has area  $\frac{1}{2} \cdot |AC| \cdot |BC| = \frac{1}{2} \cdot |BC|^2$ , so  $|BC|^2 = 200$ . Because  $|CE| = \frac{1}{2} \cdot |BC|$ , we have that  $|CE|^2 = \frac{1}{4} \cdot 200 = 50$ . Now the Pythagorean theorem gives that  $|BE|^2 = |BC|^2 + |CE|^2 = 200 + 50 = 250$ . Hence we see that  $|BE| = 5\sqrt{10}$ .

In triangle  $ABE$ , line segments  $EF$  and  $BC$  are altitudes, and since  $AG$  passes through their intersection  $D$ , line segment  $AG$  is also an altitude and is therefore perpendicular to  $BE$ . Since the area of triangle  $ABE$  is now both equal to  $\frac{1}{2} \cdot |BE| \cdot |AG| = \frac{5}{2}\sqrt{10} \cdot |AG|$  and to 150, we see that  $|AG| = 6\sqrt{10}$ .

## C-problems

- C1.** (a) In triangle  $P_0P_1P_2$ , angle  $\angle P_0P_2P_1$  is also equal to  $\alpha$  and so angle  $\angle P_0P_1P_2 = 180^\circ - 2\alpha$ . Since angle  $\angle P_0P_1P_3$  is straight, angle  $\angle P_2P_1P_3$  equals  $2\alpha$ . We then find that  $\angle P_1P_3P_2 = 2\alpha$  and  $\angle P_1P_2P_3 = 180^\circ - 4\alpha$ . Now we see that

$$\angle P_4P_2P_3 = 180^\circ - \angle P_0P_2P_1 - \angle P_1P_2P_3 = 180^\circ - \alpha - (180^\circ - 4\alpha) = 3\alpha.$$

Since triangle  $P_2P_3P_4$  is isosceles,  $\angle P_0P_4P_3 = 3\alpha$  and  $\angle P_2P_3P_4 = 180^\circ - 6\alpha$ . Finally, we now also see that

$$\angle P_0P_3P_4 = \angle P_0P_3P_2 + \angle P_2P_3P_4 = 2\alpha + (180^\circ - 6\alpha) = 180^\circ - 4\alpha.$$

So we see that triangle  $P_0P_3P_4$  is exactly isosceles if  $3\alpha = 180^\circ - 4\alpha$ , or in other words if  $\alpha = \frac{180^\circ}{7}$ .

- (b) In part (a), we saw that after 4 jumps, the base angles of the last isosceles triangle are equal to  $3\alpha$ , so the apex angle of that last isosceles triangle is  $180^\circ - 6\alpha$ . So of the straight angle at  $P_3$ , exactly  $6\alpha$  remains. The previous isosceles triangle had base angles of  $2\alpha$ , so the next isosceles triangle (if it is still possible) has base angles of  $4\alpha$ . And this fifth step can be done as long as  $4\alpha < 90^\circ$  because then we can form a new isosceles triangle with base angles  $4\alpha$ . If  $4\alpha \geq 90^\circ$  this is not possible, because an isosceles triangle cannot have obtuse or right base angles.

By repeating this argument, we see that after  $n$  jumps, the base angles of the last isosceles triangle are equal to  $(n - 1) \cdot \alpha$  and another  $(n + 1)$ -th jump can be made as long as  $n \cdot \alpha < 90^\circ$ . For  $\alpha = 6^\circ$  we have  $14 \cdot 6^\circ = 84^\circ$  so there is still a 15th jump, but  $15 \cdot 6^\circ = 90^\circ$  so no more 16th jump can be taken. So the maximum number of jumps is 15.

- C2.** (a) For  $x = 10$  the equation becomes

$$(y + 99)(y - 99) = 40y.$$

We can also write this as  $y^2 - 40y - 99^2 = 0$ . Since  $99 = 9 \cdot 11$ , we have  $99^2 = 9^2 \cdot 11^2$ . Because  $9^2 = 81$  and  $11^2 = 121$  have difference exactly 40, we see that the equation can be factored as follows:  $(y + 81)(y - 121) = 0$ . It follows that  $y = 121$  or  $y = -81$ .

- (b) We are now going to solve the equation for a general integer  $x$ . This time we can rewrite the equation as

$$y^2 - (4x)y - (x^2 - 1)^2 = 0.$$

As in part (a), we note that  $x^2 - 1 = (x - 1)(x + 1)$  hence

$$(x^2 - 1)^2 = (x - 1)^2(x + 1)^2 = (x^2 - 2x + 1)(x^2 + 2x + 1).$$

We see that the factors differ by  $4x$ . So the equation factors as

$$(y + (x - 1)^2)(y - (x + 1)^2) = 0.$$

This means that either  $y = -(x - 1)^2$  or  $y = (x + 1)^2$ . In all cases either  $y$  or  $-y$  is the square of an integer.