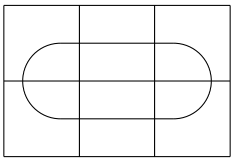
Second round
Dutch Mathematical Olympiad

Friday 11 March 2022

Solutions

B-problems

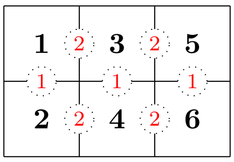
B1. 11 For two adjacent squares, the difference between the numbers in these squares is called the *border score* of the common side. We consider the following way to walk around the rectangle.



We first consider what happens when you start in the square with the 1 and walk to the square with the 6 following this circuit. If the numbers that you encounter during this walk are increasing, then the border scores that you encounter on your way add up to 5. If you also encounter a number that is smaller than the previous number, then the total border score on the way from 1 to 6 will be greater than 5.

For the rest of the circuit, going from the square with the 6 back to the square with the 1, the same holds: the border scores that you encounter on your way add up to at least 5. Hence, for the whole circuit the total border score is at least 10. The only border score that we did not count yet is the border score between the middle two squares. This border score is at least 1. Hence, the total score is at least 11.

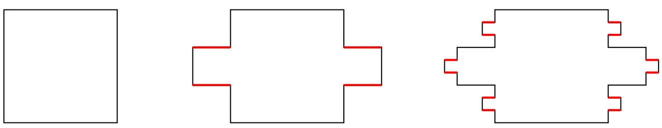
Below you see a distribution with score 11 which shows that 11 is the smallest possible score.



B2. 39 If k is an even integer, then k^2 is even so it can never be of the form $8n+1$. If k is odd, say $k = 2\ell + 1$, then we have $k^2 = (2\ell+1)^2 = 4\ell^2 + 4\ell + 1 = 4\ell(\ell+1) + 1$. One of the numbers ℓ or $\ell + 1$ is even, so $4\ell(\ell+1)$ is divisible by 8 and k^2 is of the form $8n+1$.

Furthermore we see that $8n+1$ is at least 9, the square of 3, and at most $8 \cdot 800 + 1 = 6401$. What is left is to find out how many odd squares are in between. We see that $80^2 = 6400$ is smaller than 6401 and that $81^2 = 6561$ is greater than 6401. Thus the squares we are looking for are $3^2, 5^2, 7^2, \ldots, 79^2$. These are 39 squares in total.

B3. 84 The circumference of the figure consists of horizontal and vertical line segments. When a new small square grows, the vertical line segment is split up, and the middle part moves to the side. The total length of the vertical line segments stays the same. We only have to figure out which extra horizontal pieces we get; in the figure below, these new segments are coloured red.



During the first minute four pieces of length $\frac{1}{3}$ are created; together that is $\frac{4}{3}$. During the second minute, 12 new pieces of length $\frac{1}{9}$ are created; that is also $\frac{4}{3}$ together.

Every minute, every vertical piece is split into three pieces, which means that during the next minute three times as many new squares grow. So during each minute the number of new segments is three times as large as the previous minute, but they also have a length which is $\frac{1}{3}$ of the length they had during the previous minute. We conclude that the circumference of the figure grows with $\frac{4}{3}$ every minute. Hence, after 60 minutes, the circumference is $4 + 60 \cdot \frac{4}{3} = 84$.

B4. 20 Suppose Lavinia buys a gold boxes, b silver, and c bronze. Then we can describe the problem we want to solve with the following equations:

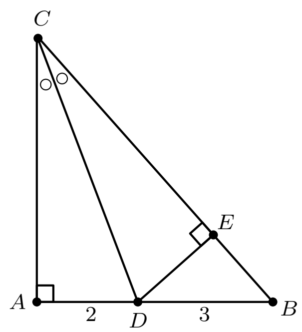
$$2a + b + 5c = 3a + 2b + c = a + 4b + 2c.$$

The first equation $2a + b + 5c = 3a + 2b + c$ can be rewritten to $4c - b = a$. If we substitute this expression for a in the second equation, we find

$$13c - b = 3b + 6c.$$

This can be simplified to $7c = 4b$. The number 7 is prime and this means b has to be divisible by 7 (since 4 is not divisible by 7). We write $b = 7k$, with k an integer. This implies $c = \frac{4}{7}b = 4k$ and $a = 4 \cdot 4k - 7k = 9k$. We see that a, b, and c are as small as possible if k is as small as possible. Lavinia buys at least one box of chocolates, so it has to be that $k = 1$ and it follows that $a = 9$, $b = 7$, and $c = 4$. We see that Lavinia has 45 chocolates of each flavour. In total she has bought at least $9 + 7 + 4 = 20$ boxes.

B5. $2\sqrt{6}$ We draw a line perpendicular to BC through D; the projection of D onto BC is called E. We have $\angle ACD = \angle ECD$. Moreover, we have right angles $\angle DAC = \angle DEC$. The triangles ACD and ECD have the side CD in common and two equal angles, hence the triangles ACD and ECD are each other's reflections in CD. This yields $|DE| = |DA| = 2$ and $|AC| = |EC|$.



Now we will apply the Pythagorean theorem three times. We first do that in triangle DEB, which yields $2^2 + |EB|^2 = 3^2$, or $|EB| = \sqrt{9-4} = \sqrt{5}$. Now let $x = |AC| = |EC|$. By applying the Pythagorean theorem to triangle ABC, we obtain

$$x^2 + (2+3)^2 = (\sqrt{5} + x)^2, \quad \text{so} \quad x^2 + 25 = 5 + 2\sqrt{5}x + x^2, \quad \text{so} \quad 20 = 2\sqrt{5}x.$$

We get that $x = 2\sqrt{5}$. Finally we apply the Pythagorean theorem to triangle CDE to get $2^2 + (2\sqrt{5})^2 = |CD|^2$, hence $|CD| = \sqrt{24} = 2\sqrt{6}$.

C-problems

C1. We first note that there is a useful relation between a, b, d, and v. The total number of integers on the two pieces of paper is $a + b$, the number of integers on Alicia's piece of paper plus the number of integers on Britt's piece of paper. This, however, also equals $v + d$: the total number of distinct integers, plus the total number of integers that have been written down twice. Hence, we get that $a + b = v + d$.

(a) We choose $a = b = 2022$ and look for a solution to $a \cdot b = d \cdot (v + d)$. We use the fact that $a + b = v + d$. This means that we are looking for solutions to $a \cdot b = d \cdot (a + b)$. If we substitute $a = b = 2022$, then we find that $2022 \cdot 2022 = d(2022 + 2022) = d \cdot 2 \cdot 2022$, so $d = 1011$. Together with $a + b = v + d$, we find that $2022 + 2022 = v + 1011$, so $v = 3033$. This situation happens for example if Alicia writes down the numbers 1 to 2022, and Britt writes down the numbers 1012 to 3033. □

(b) With a little bit of trying, and by choosing d not too large, we find that $a = b = 3$, $d = 1$, and $v = 5$ is a solution: $3 \cdot 3 = 1 \cdot (5+4)$. The numbers also satisfy the equation $a+b = v+d$. This situation can occur if Alicia writes down the numbers 1, 2, and 3, and Britt writes down the numbers 3, 4, and 5, for example. □

(c) Suppose that there are numbers such that $a \cdot b = v \cdot d$. We already deduced that $a+b = v+d$, or $v = a + b - d$. Substituting this yields

$$ab = vd = (a + b - d)d = ad + bd - d^2.$$

If we now subtract ad from both sides of this equation, we find $ab - ad = bd - d^2$, so $a(b - d) = d(b - d)$. Because Britt wrote down at least one number that Alicia did not write down, we have $b > d$. Therefore, we can divide the equation $a(b - d) = d(b - d)$ by the positive number $b - d$, and we find that $a = d$. On the other hand, Alicia wrote down at least one number that Britt did not write down, hence $a > d$. This gives a contradiction and hence there cannot exist numbers such that $a \cdot b = v \cdot d$. □

C2. (a) First we look at the last two digits of a sunny number. There are nine possibilities for these: 01, 12, 23, 34, 45, 56, 67, 78, and 89. If we then look at twice a sunny number, we get the following nine possibilities, respectively, for the last two digits: 02, 24, 46, 68, 90, 12, 34, 56, and 78. We see that twice a number can only be sunny if the original sunny number ends in 56, 67, 78, or 89. In all four cases we see that by doubling a 1 carries over to the hundreds.

Now we look at the first two digits of a sunny number. The nine possibilities are 10, 21, 32, 43, 54, 65, 76, 87, and 98. If the first digit is 5 or higher, twice the number has more than four digits so it can never be sunny. The possibilities 10, 21, 32, and 43 are left. After doubling and adding the carried over 1 to the hundreds we get, respectively, 21, 43, 65, and 87. In all cases twice a sunny number is a sunny number if the first digits of the original sunny number are 10, 21, 32, or 43 and the last two digits are 56, 67, 78, or 89. In total there are $4 \cdot 4 = 16$ combinations to be made, hence 16 sunny numbers for which twice the number is again sunny. □

(b) Denote by a and b the two middle digits of a sunny number. Then the two digits on the outside are $a+1$ and $b+1$, so the number is $1000(a+1)+100a+10b+(b+1) = 1100a+11b+1001$. This number is divisible by 11 because $1100a$ as well as $11b$ as well as $1001 = 91 \cdot 11$ is divisible by 11. After division by 11 we get the number $100a + b + 91$. Now b is at most 8, because $b + 1$ has to be a digit as well. Furthermore a is at least 1, because the number we started with has to be at least 2000. So we see that $100a + b + 91 = 100a + 10 \cdot 9 + (b + 1)$ is the three-digit number with digits a, 9, and $b + 1$, a three-digit number with a 9 in the middle. □