Second round
Dutch Mathematical Olympiad

Friday 12 March 2021

Solutions

B-problems

B1. 2020 We consider a small piece of the three sequences.

$$\begin{array}{cccccccc}
\dots & a & b & c & d & \dots & \dots & \dots \\
\dots & \dots & a & b & c & d & \dots & \dots \\
\dots & \dots & \dots & a & b & c & d & \dots
\end{array}$$

In the middle two columns, we see that if we add the numbers in the column with d, c, and b, we get 3 more than when we add the numbers in the column with c, b, and a. Hence, we get $(d+c+b)-(c+b+a)=3$, or $d-a=3$. This means that when we move three places to the right in the uppermost sequence, the number grows by 3.

We are looking for the 2021st number in the sequence. Because the remainder of 2021 upon division by 3 is 2, we only need to consider the numbers that occur on places whose number has a remainder of 2 upon division by 3. The second number is 1, the fifth number (three places to the right) is $1+3=4$, etcetera. This sequence is:

$$\dots, 1, \dots, \dots, 4, \dots, \dots, 7, \dots, \dots, 10, \dots, \dots, 13, \dots$$

In general, we see that for an n which has remainder 2 upon division by 3, the n-th number in the sequence equals $n-1$. Hence, the 2021st number is 2020.

B2. 50 We consider an arbitrary combi number. We will first prove that this number consists of at least 50 digits. Then we will show that there exists a 50 digit combi number.

Note that if we swap two digits everywhere in a combi number, we still get a combi number: you can replace all threes by ones and vice versa, for example. In this way, a combi number never has to start with a zero.

Suppose that the digit c does not occur on the first or last position of the combi number. Each time that c occurs in the combi number, it forms a pair with both its left and right neighbour. Because c has to occur in 9 pairs in total, the digit c must occur (at least) 5 times in the combi number.

Suppose the digit c is the first or last digit, but not both. Then c must also occur 5 times in the combi number, because the c at the start (or end) only occurs in one pair. If both first and last digit of a combi number are the same digit c, then you need six copies of the digit c.

Each digit from 0 to 9 has to occur at least five times in the combi number. Hence, the combi number has at least 50 digits in total. There also exists a 50 digit combi number, for example the number

$$98765432109753196307418529517394864208406283726150.$$

The way we constructed this number is as follows. We took all possible sequences of digits with the same difference. For example, we took 98765432109 *where we jump down with 1,* 975319 *and* 864208 *with a jump size of 2, etcetera. Then we glued these sequences in one big sequence in a clever way. There are of course many other ways to construct a 50 digit combi number.*

B3. 15 We look at a rectangle that is $n \geqslant 5$ small rectangles high, and count the squares. Because the small rectangles have height 2, we can only find squares whose side has even length. The smallest squares that can appear, are 2×2-squares. We see that there are 9 of them in every row. Hence there are $9n$ of these squares in total.

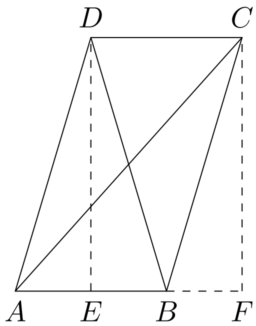
The second smallest squares that can appear, are 4×4-squares. Such a square covers two adjacent rows. These can be rows 1 and 2, rows 2 and 3, etcetera, up until row $n-1$ and n. This gives $n-1$ possibilities for the rows. If we now look at how the squares cover four adjacent columns, we see there are 7 possibilities. One way to see that is by looking at the left most column of such a set of four adjacent columns. The possibilities for this column are columns 1 up until 7 (if you start with column 8, you end outside the figure). So, in total there are $7(n-1)$ squares with side length 4 in the figure.

In the same way we can count the 6×6-squares: there are $5(n-2)$ of them in total. There are $3(n-3)$ squares with side length 8, and $1(n-4)$ squares with side length 10. Bigger squares cannot appear in the figure. Hence in total there are

$$9n + 7(n-1) + 5(n-2) + 3(n-3) + (n-4) = 25n - 30$$

squares in the figure. If we solve the equation $345 = 25n - 30$, we find the solution $n = \frac{375}{25} = 15$. In the beginning we assumed that $n \geqslant 5$. We made this assumption because otherwise there are no 10×10-squares in the figure and the formulas above might not be correct. However, in the case that $n < 5$ there are of course fewer squares in the figure than in the case $n \geqslant 5$. It follows that $n = 15$ is the only solution.

B4. 9 We first draw some extra lines, see the figure. The parallelogram is $ABCD$ with $|AB| = 4$ and $|AD| = |BC| = 7$. We are now trying to find $|AC|$.



Because $|AD| = |BD|$, the triangle $\triangle ABD$ is isosceles. Let E be the midpoint of A and B. Then triangles $\triangle AED$ and $\triangle BED$ are congruent and angle $\angle AED$ is a right angle.

From point C we orthogonally project onto the line AB, and we let F be the intersection. Then $\angle DAE = \angle CBF$ because AD and BC are parallel, and $\angle AED = \angle BFC = 90°$. Moreover, we have $|AD| = |BC|$, hence we get the congruency $\triangle AED \cong \triangle BFC$. Hence, we have $|BF| = 2$.

We now apply the Pythagorean theorem to $\triangle BFC$. For the height h of this triangle, we get that $2^2 + h^2 = 7^2$, hence $h^2 = 45$. Applying the Pythagorean theorem to triangle $\triangle AFC$ in order to find the diagonal $d = |AC|$ yields: $h^2 + 6^2 = d^2$, or $45 + 36 = d^2$, hence $d = 9$.

B5. 57 Consider the situation where we add a fourth wheel to the row with circumference 2 cm. The small wheel turns the fastest of all. If the wheel of 14 cm makes one lap, the fourth wheel makes 7 laps. One lap of the wheel of 10 cm corresponds to 5 laps of the fourth wheel, and one lap of the wheel of 6 cm corresponds to 3 laps of the small wheel.

The total number of laps that the small wheel has to make before we get back to the starting configuration is a multiple of 3, 5 and 7. The smallest number that is a multiple of 3, 5 and 7 is 105. So in total the wheels make, from big to small, 15, 21 and 35 rounds. This means that there are at most $15 + 21 + 35 = 71$ whistles, but there might be fewer, because two arrows could point up simultaneously.

Exactly halfway the 105 laps of the fourth wheel the other wheels have made $7\frac{1}{2}$, $10\frac{1}{2}$ and $17\frac{1}{2}$ laps. This means all three arrows point up. This gives 2 whistles we counted too much.

Now we will investigate how often it happens that the arrows on the two biggest wheels point up at the same time. From the situation we had exactly halfway, the fourth wheel has to turn a multiple of 5 and a multiple of 7 laps, in either direction. This situation appears after $52\frac{1}{2} - 35$ and $52\frac{1}{2} + 35$ laps of the fourth wheel, counted from the start. In this situations the arrow on the third wheel does not point up, because 35 is not a multiple of 3.

In a similar way we find that the arrows on the first and the third wheel both point up after $52\frac{1}{2} - 42$, $52\frac{1}{2} - 21$, $52\frac{1}{2} + 21$ and $52\frac{1}{2} + 42$ laps of the fourth wheel. In these situations the arrow on the second wheel does not point up, because 21 and 42 are not multiples of 5.

Finally, the arrows on the second and third wheel both point up after $52\frac{1}{2} - 45$, $52\frac{1}{2} - 30$, $52\frac{1}{2} - 15$, $52\frac{1}{2} + 15$, $52\frac{1}{2} + 30$ and $52\frac{1}{2} + 45$ rounds of the fourth wheel. In these situations the arrow on the first wheel does not point up, because 15, 30 and 45 are not multiples of 7.

Together we find there are $2 + 4 + 6 = 12$ situations where exactly two of the arrows point up. This gives another 12 whistles we have counted too much. Thus in total we hear $71 - 2 - 12 = 57$ whistles.

C-problems

C1. (a) Consider the situation where the first player has 2 coins, the second player has 0 coins and all other players have 1 coin. This situation looks as follows:

$$2\,0\underbrace{1\,1\cdots 1\,1}_{n-2 \text{ ones}}$$

For example, for $n = 3$ the starting distribution is 201. We see that the first and the third player both give a coin to the second player. This gives the distribution 120. This is exactly the distribution 201 if you shift all players by one place. We see that the game never stops.
In this case the first player has to give a coin to the second player, the third player has to give a coin to the left and all other players keep their coin. We end with the following situation.

$$1\,2\,0\underbrace{1\,1\cdots 1\,1}_{n-3 \text{ ones}}$$

This is exactly the same distribution as the starting distribution, except now it is player 2 that has 2 coins and player 3 that has 0 coins. If we continue playing, there will always be a player with 2 coins and thus the game never stops. □

(b) For $n \geqslant 4$ we can consider the following starting distribution.

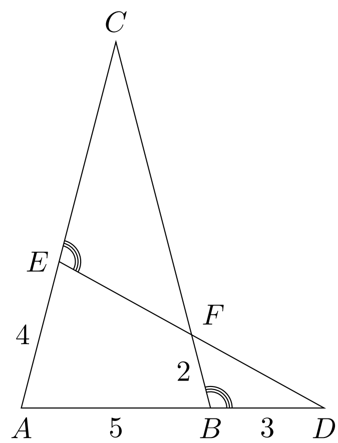
$$2\,0\,0\,2\underbrace{1\,1\cdots 1\,1}_{n-4 \text{ ones}}$$

For example, for $n = 4$ the starting distribution is 2002. In this case there is no player with exactly one coin. The first and the last player give a coin to the second and third player, respectively. Then the game stops.
The first player has to give a coin to the right and the fourth player has to give a coin to the left. All other players keep their coin. This gives a situation where all players have 1 coin, thus the game stops. □

C2. (a) The situation is depicted in the figure on the right. Because angles $\angle AEC$ and $\angle ABD$ are straight, we have

$$\angle ABC = 180^\circ - \angle DBC = 180^\circ - \angle DEC = \angle AED.$$

Because angle A occurs in both triangles, triangles $\triangle ABC$ and $\triangle AED$ have two equal angles, and hence the triangles are similar. □



(b) Because of the similarity of triangles $\triangle ABC$ and $\triangle AED$, the angles at C and D are equal. Together with the equality $\angle DBF = \angle CEF$, it follows that triangles $\triangle DBF$ and $\triangle CEF$ are similar.

In a pair of similar triangles, all pairs of sides have the same ratio. Hence, the similarity of triangles $\triangle DBF$ and $\triangle CEF$ yields

$$\frac{|BF|}{|EF|} = \frac{|FD|}{|FC|} = \frac{|DB|}{|CE|}. \tag{1}$$

As triangles $\triangle ABC$ and $\triangle AED$ are similar, we find that

$$\frac{|AB|}{|AE|} = \frac{|BC|}{|ED|} = \frac{|CA|}{|DA|}. \tag{2}$$

Using equations (1) and (2), we can now find $|CF|$. Using the first and last ratio in equation (2), we get

$$\frac{5}{4} = \frac{|AB|}{|AE|} = \frac{|AC|}{|AD|} = \frac{4 + |EC|}{5 + 3}.$$

Hence, we have $|EC| = 6$. If we substitute in the first and third ratio in equation (1), we get $\frac{2}{|EF|} = \frac{3}{6}$. Hence, we have $|EF| = 4$. Using the first and second ratio in (1), we now get that $\frac{2}{4} = \frac{|FD|}{|FC|}$ hence $|FD| = \frac{1}{2}|CF|$. Finally, we substitute this in the first and second ratio in equation (2):

$$\frac{5}{4} = \frac{|AB|}{|AE|} = \frac{|BC|}{|DE|} = \frac{2 + |CF|}{4 + \frac{1}{2}|CF|}.$$

Taking cross ratios and solving the remaining equation, we get $|CF| = 8$. □