Second round
Dutch Mathematical Olympiad

Friday 13 March 2020

Solutions

B-problems

B1. 49999 We denote the digit sum of a number n by $S(n)$. We are looking for the smallest positive integer n having the following property:

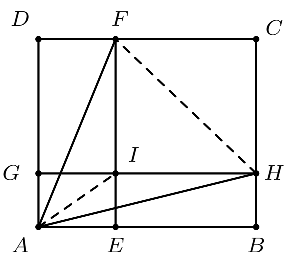
- $S(n)$ and $S(n+1)$ are both divisible by 5.

The last digit of n must be a 9. If it were not, then $S(n+1)$ would equal $S(n)+1$, because the last digit of $n+1$ would be one greater than the last digit of n and the other digits are the same. However, $S(n)$ and $S(n)+1$ cannot both be divisible by 5, as this would cause their difference $S(n+1)-S(n)=1$ to also be divisible by 5.

Hence, the number n ends with one or more times the digit 9; say it ends with k times the digit 9. If we add 1 to n, then each of these k digits 9 changes into the digit 0 and the digit before the nines is raised by 1 (if n only consists of digits 9, then $n+1$ consists of the digit 1 followed by k zeros). Hence, we have $S(n+1)=S(n)-9\cdot k+1$. Because $S(n+1)$ and $S(n)$ are both divisible by 5, also their difference $S(n)-S(n+1)=9k-1$ is divisible by 5. The smallest k for which $9k-1$ is divisible by 5, is $k=4$.

Because $S(9999)=36$ is not divisible by 5, we will try the five digit numbers ending with four digits 9 next. The first digit of n is denoted by a (hence n is written as $a9999$). The digit sum $S(n)=a+4\cdot 9=a+36$ must be divisible by 5. The smallest a for which this is the case, is $a=4$. As $S(49999)=40$ and $S(50000)$ are both divisible by 5, the number $n=49999$ has the required property. Hence, it is the smallest number having the property.

B2. $24\frac{1}{5}$ Instead of considering the whole parallelogram $AHJF$, we first consider half of it: triangle AFH. We can subdivide AFH into three smaller triangles: AFI, FIH, and HIA. If we compute the area of triangle AFI using the formula $\frac{1}{2}\cdot\text{base}\cdot\text{height}$ and take FI as base, then we see that it is exactly half of the area of rectangle $DFIG$. The area of triangle FIH is half of the area of rectangle $CHIF$, and the area of triangle HIA is half of the area of rectangle $BHIE$.

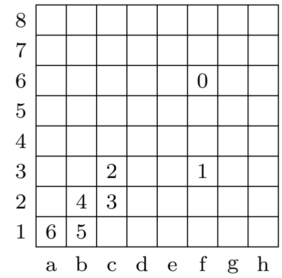


So the total area of the parallelogram, two times the area of triangle AFH, equals the sum of the areas of the rectangles $DFIG$, $CHIF$, and $BHIE$. The last two areas are 12 and 5. So we only have to find the area of $DFIG$. This area equals

$$|GI|\cdot|FI|=\frac{(|GI|\cdot|IE|)\cdot(|FI|\cdot|IH|)}{|IH|\cdot|IE|}=\frac{3\cdot 12}{5}=7\tfrac{1}{5}.$$

Hence, the total area of $AHJF$ is $5+12+7\frac{1}{5}=24\frac{1}{5}$.

B3. 43 In the figure on the right the positions where the square f6 lands during the folding are indicated. We start at f6 (after 0 folds). After that, our square lands on positions f3, c3, c2, b2, b1, and a1.



Each time we fold, the order of the squares in the stack is reversed: the squares that were under f6 get on top of f6, and vice versa. Moreover, the stack lands on top of another stack of equal height. After k folds, the stack consists of 2^k squares. If at that point there are a squares on top of f6 and b squares below f6, then after $k+1$ folds there are b squares on top of f6 and $a+2^k$ squares below f6. So we made the following table.

Number of folds	0	1	2	3	4	5	6
Squares in stack	1	2	4	8	16	32	64
Squares on top of f6	0	0	1	2	5	10	21
Squares below f6	0	0+1=1	0+2=2	1+4=5	2+8=10	5+16=21	10+32=42

After six folds, there are 42 squares below f6, so f6 gets number 43.

B4. 100 We number the brownies around the circle. The brownie having the smallest number of chestnuts gets number 1. Her left neighbour is brownie 2, the left neighbour of brownie 2 is brownie 3, and so on, until brownie 100, who is the right neighbour of brownie 1. The number of chestnuts that brownie k has, is denoted by a_k.

We now consider the two neighbours of brownie 1. The right neighbour has a_{100} chestnuts. Because $a_1 < a_{100}$, the division of a_1 by a_{100} gives remainder a_1. This remainder a_1 is written on the green piece of paper of brownie 1. The left neighbour of brownie 1 has got a_2 chestnuts. Because $a_2 > a_1$, the division of a_2 by a_1 gives a remainder r, which is smaller than a_1. This remainder r is written on the green piece of paper of brownie 2. So the two numbers occurring on the one hundred green pieces of paper are a_1 and r.

We now consider brownies 1 to 99. We will prove that $a_1 < a_2 < \cdots < a_{100}$. Suppose that this were not true. For a certain k, we would have $a_{k+1} < a_k$ (there cannot be equality as everyone has a different number of chestnuts). The remainder of a_{k+1} upon division by a_k would be a_{k+1} and that would be the number on the green piece of paper of brownie $k+1$. This number must equal a_1 or r. But this is impossible, because $a_1 < a_{k+1}$ (brownie 1 is having the smallest number of chestnuts) and r is even smaller than a_1. We conclude that we indeed must have $a_1 < a_2 < \cdots < a_{100}$.

If we now consider the numbers on the red pieces of paper of brownies 1 to 99, then we see that these are the numbers a_1 to a_{99}. The number on the red piece of paper of brownie 100 is the remainder of a_{100} upon division by a_1. This remainder is smaller than a_1, so the one hundred numbers on the red pieces of paper are all distinct.

B5. 02 We observe that $a_{2000} = a_{1999} \cdot 2000$ ends in three zeros. We can now determine the last two digits of a_{2001} to a_{2020} using the given formulas. To determine the last two digits of a number, we only need to know the last two digits of the previous number. Therefore, we will only keep track of the last two digits of a_{2001} to a_{2020}.

a_{2001}	a_{2002}	a_{2003}	a_{2004}	a_{2005}	a_{2006}	a_{2007}
$0+1=1$	$1+2=3$	$3+3=6$	$6+4=10$	$10+5=15$	$15+6=21$	$21+7=28$

a_{2008}	a_{2009}	a_{2010}	a_{2011}	a_{2012}	a_{2013}	a_{2014}
$28 \cdot 8 = 224$	$24+9=33$	$33+10=43$	$43+11=54$	$54+12=66$	$66+13=79$	$79+14=93$

a_{2015}	a_{2016}	a_{2017}	a_{2018}	a_{2019}	a_{2020}	
$93+15=108$	$8 \cdot 16 = 128$	$28+17=45$	$45+18=63$	$63+19=82$	$82+20=102$	

We find that the last two digits of a_{2020} are 02.

C-problems

C1. (a) We observe that $(n+1)! = (n+1)\cdot n!$, and therefore that $n!\cdot(n+1)! = (n!)^2\cdot(n+1)$. That product is a perfect square if and only if $n+1$ is a perfect square, since $(n!)^2$ is a perfect square. For $1 \leqslant n \leqslant 100$ this is the case for $n = 3, 8, 15, 24, 35, 48, 63, 80, 99$ (perfect squares minus one that are below 100). □

(b) We rewrite the product $n!\cdot(n+1)!\cdot(n+2)!\cdot(n+3)!$ as follows:

$$(n!)^2\cdot(n+1)\cdot(n+2)!\cdot(n+3)! = (n!)^2\cdot(n+1)\cdot((n+2)!)^2\cdot(n+3).$$

Since $(n!)^2$ and $((n+2)!)^2$ are both perfect squares, the above product is a perfect square if and only if $(n+1)(n+3)$ is a perfect square. However, $(n+1)(n+3)$ cannot be a perfect square. Indeed, suppose that $(n+1)(n+3) = k^2$ were a perfect square. Since $(n+1)^2 < (n+1)(n+3) < (n+3)^2$ we would have $n+1 < k < n+3$, so $k = n+2$. This is impossible because $(n+1)(n+3) = (n+2)^2 - 1$, which is not equal to $(n+2)^2$. □

C2. An octagon can be subdivided into six triangles (see figure on the left). Together, the angles of those six triangles add up to the same number of degrees as the eight angles of the octagon. Since the angles of any triangle add up to 180 degrees, this means that the eight angles of the octagon add up to $6\cdot 180° = 1080°$. Hence, each of the angles of the regular octagon is $\frac{1}{8}\cdot 1080° = 135°$.

We now consider the figure from the problem statement (see figure on the right). Line segment BP bisects angle ABC, so $\angle ABP = \angle PBC = 67\frac{1}{2}°$. Since triangles ABP and BCP are isosceles (as $|AB| = |AP|$ and $|BC| = |CP|$), we also have $\angle APB = \angle BPC = 67\frac{1}{2}°$ and $\angle BAP = \angle BCP = 180° - 135° = 45°$.

In triangles ABQ and BCR all sides have the same length. These triangles are therefore equilateral and all angles are $60°$. From this, we deduce that $\angle PAQ = \angle BAQ - \angle BAP = 15°$. In the same way, we find $\angle PCR = 15°$. Furthermore, triangles PAQ and PCR are isosceles (since $|AP| = |AQ|$ and $|CP| = |CR|$), so $\angle APQ = \frac{1}{2}(180° - 15°) = 82\frac{1}{2}°$ and $\angle CPR = 82\frac{1}{2}°$.

By mirror symmetry, PQ and PR have the same length, so PQR is an isosceles triangle with apex P. We have already determined all angles at P, except $\angle QPR$. We deduce that

$$\angle QPR = 360° - \angle APQ - \angle APB - \angle BPC - \angle CPR = 360° - 2\cdot 67\tfrac{1}{2}° - 2\cdot 82\tfrac{1}{2}° = 60°.$$

From this and the fact that PQR is isosceles, we directly conclude that PQR is equilateral. □