Second round
Dutch Mathematical Olympiad

Friday 13 March 2020

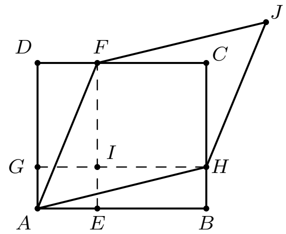
- Time available: 2.5 hours.
- The competition consists of five B-problems and two C-problems.
- Formula sheets and calculators are not allowed. You can only use a pen, compass, ruler, set square, and of course your mental skills.
- Good luck!

B-problems

For the B-problems you only have to give the answer (for example, a number). No explanation is required. A correct answer is awarded 4 points. For a wrong or incomplete answer no points are given. Please work very accurately: a minor error in a calculation may result in a wrong answer.
NOTE: All answers should be given in exact and reduced form, like $\frac{11}{81}$, or $2+\frac{1}{2}\sqrt{5}$, or $\frac{1}{4}\pi+1$, or 3^{100}.

B1. The *digit sum* of a number is obtained by adding all digits of the number. For example, the digit sum of 1303 is $1+3+0+3=7$.
Find the smallest positive integer n for which both the digit sum of n and the digit sum of $n+1$ are divisible by 5.

B2. Rectangle $ABCD$ is subdivided into four rectangles as in the figure. The area of rectangle $AEIG$ is 3, the area of rectangle $EBHI$ is 5, and the area of rectangle $IHCF$ is 12.
What is the area of the parallelogram $AHJF$?



B3. A square sheet of paper lying on the table is divided into $8\times 8=64$ equal squares. These squares are numbered from **a1** to **h8** as on a chess board (see fig. 1). We now start folding, in such a way that square **a1** always stays in the same spot on the table. First we fold along the horizontal midline (fig. 1). This will cause square **a8** to fold on top of square **a1**. Then we fold along the vertical midline (fig. 2). Next, we fold along the new horizontal midline (fig. 3), et cetera. After folding six times, we have a small package of paper in front of us (fig. 7) that we can consider as a stack of 64 square pieces of paper.

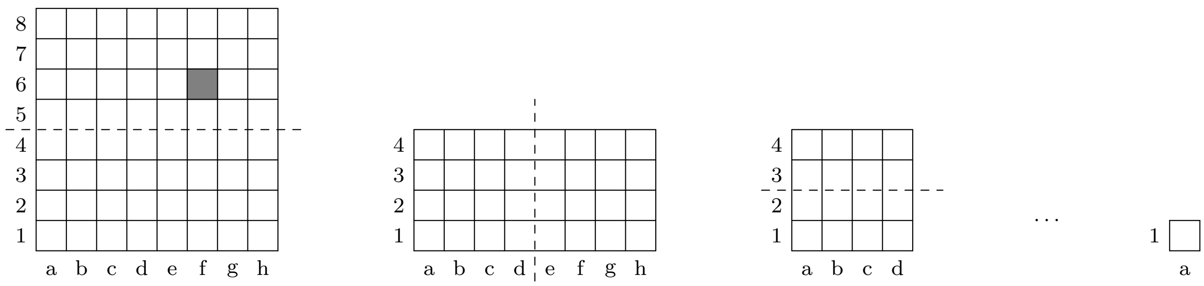


fig. 1 fig. 2 fig. 3 fig. 7

The squares in this stack are numbered from bottom to top from 1 to 64. So square **a1** gets number 1.
Which number does square **f6** get?

PLEASE CONTINUE ON THE OTHER SIDE

B4. One hundred brownies (girl scouts) are sitting in a big circle around the camp fire. Each brownie has one or more chestnuts and no two brownies have the same number of chestnuts. Each brownie divides her number of chestnuts by the number of chestnuts of her *right* neighbour and writes down the remainder on a green piece of paper. Each brownie also divides her number by the number of chestnuts of her *left* neighbour and writes down the remainder on a red piece of paper. For example, if Anja has 23 chestnuts and her right neighbour Bregje has 5, then Anja writes 3 on her green piece of paper and Bregje writes 5 on her red piece of paper.

If the number of distinct remainders on the 100 green pieces of paper equals 2, what is the smallest possible number of distinct remainders on the 100 red pieces of paper?

B5. Given is the sequence of numbers $a_0, a_1, a_2, \ldots, a_{2020}$ with $a_0 = 0$. Furthermore, the following holds for every $k = 1, 2, \ldots, 2020$:

$$a_k = \begin{cases} a_{k-1} \cdot k & \text{if } k \text{ is divisible by 8,} \\ a_{k-1} + k & \text{if } k \text{ is } \textit{not} \text{ divisible by 8.} \end{cases}$$

What are the last two digits of a_{2020}?

C-problems

For the C-problems not only the answer is important; you also have to write down a clear reasoning that shows the correctness of your answer. A correct and well-explained answer is awarded 10 points. Partial solutions may also be worth some points. Therefore, write everything down clearly and hand in your drafts.

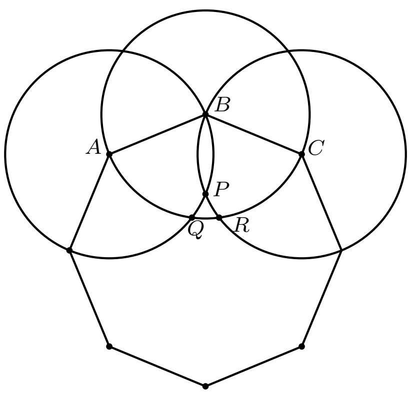
ATTENTION: Use separate sheets of paper for each C-problem and also hand in the drafts for each problem separately.

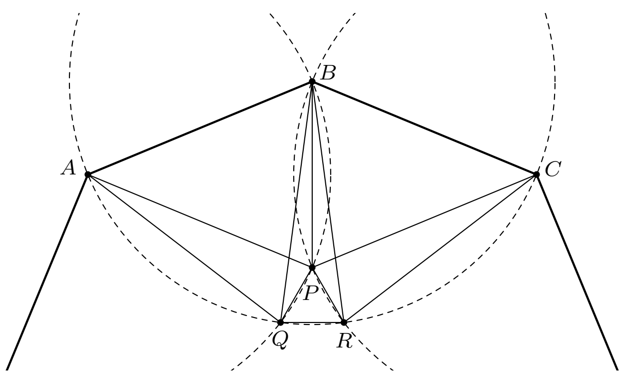
C1. Given a positive integer n, we denote by $n!$ ('n factorial') the number we get if we multiply all integers from 1 to n. For example: $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

(a) Determine all integers n with $1 \leqslant n \leqslant 100$ for which $n! \cdot (n+1)!$ is a perfect square. *Also, prove that you have found all solutions n.*

(b) Prove that no positive integer n exists such that $n! \cdot (n+1)! \cdot (n+2)! \cdot (n+3)!$ is a perfect square.

C2. Three consecutive vertices A, B, and C of a regular octagon (8-gon) are the centres of circles that pass through neighbouring vertices of the octagon. The intersection points P, Q, and R of the three circles form a triangle (see figure).





Prove that triangle PQR is equilateral.