Second round
Dutch Mathematical Olympiad

Friday 15 March 2019

Solutions

B-problems

1. 42 Mother leaves the house 10 minutes after Anna and Birgit's departure. After riding for 10 minutes, she catches up with Anna. Indeed, by that time Anna has cycled for $10 + 10$ minutes, but is going half as fast as mother. After 10 minutes riding back, mother passes the house on her way to Birgit. After another 6 minutes she catches up with Birgit. Indeed, by that time Birgit has walked for $10 + 10 + 10 + 6 = 36$ minutes, while she is going six times as slow as mother. After another 6 minutes, mother is back home. This is exactly $10 + 10 + 10 + 6 + 6 = 42$ minutes after Anna and Birgit's departure.

2. 11 Suppose we have taken 11 notes with the numbers $a_1 < a_2 < \cdots < a_{11}$ such that no three meet our wish. First, we know that $a_1 \geqslant 1$ and $a_2 \geqslant 2$. Because the three notes a_1, a_2, and a_3 do not meet our wish, we must have that $a_3 \geqslant a_1 + a_2 \geqslant 1 + 2 = 3$. The three notes a_2, a_3, and a_4 also don't meet our wish, which implies that $a_4 \geqslant a_2 + a_3 \geqslant 2 + 3 = 5$. Continuing like this, we deduce

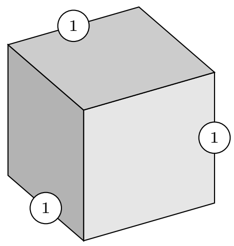
$$a_5 \geqslant 3 + 5 = 8, \qquad a_6 \geqslant 5 + 8 = 13, \qquad a_7 \geqslant 8 + 13 = 21, \qquad a_8 \geqslant 13 + 21 = 34,$$

$$a_9 \geqslant 21 + 34 = 55, \qquad a_{10} \geqslant 34 + 55 = 89, \qquad \text{and} \qquad a_{11} \geqslant 55 + 89 = 144.$$

However, the greatest number occurring on our notes is 100. Therefore, this situation cannot arise. We conclude that after taking 11 notes, there must always be three notes meeting our wish.

Moreover, it does not suffice to take 10 notes. Indeed, if we take the notes 1, 2, 3, 5, 8, 13, 21, 34, 55, and 89, then there are no three notes meeting our wish.

3. -12 We shall first show that the outcome is always at least -12. In case we write -1 on each of the twelve edges, each face gets a 1. In this case, the outcome is $-12 + 6 = -6$. For every -1 on an edge that we change into a 1, at most two faces with a 1 change into a -1. The outcome will thus be at most 2 smaller ($-1 + 1 + 1$ becomes $1 - 1 - 1$). To get below -12, we must write a 1 on at least 4 of the edges. However, in this case the outcome is at least $(4 - 8) - 6 = -10$. Hence, the outcome will never be smaller than -12.



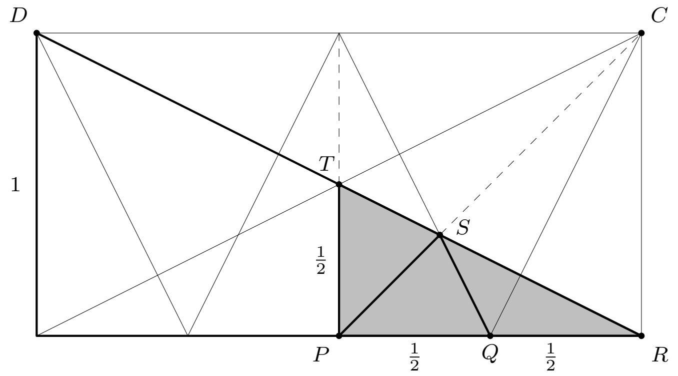
Finally, we show that -12 can be obtained as an outcome. To get this, we write a -1 on each of the edges, except for the three edges indicated in the figure. Each face has exactly three edges with a -1. Hence, each face will have a -1 as well. As a result, the outcome is $(3 - 9) - 6 = -12$.

4. 666 and 1999 We consider two cases separately: n is *even* and n is *odd*.

First, suppose n is *even*. We write $n = 2k$ for a certain integer k. We then get: $n^2 = 4k^2$ and $2n + 1 = 4k + 1$. If we divide $4k^2$ by $4k + 1$, then the outcome is $k - 1$ and the remainder is $4k^2 - (k - 1)(4k + 1) = 3k + 1$. Because the remainder must be 1000, we find $3k + 1 = 1000$ and therefore $k = 333$. This yields the solution $n = 2 \cdot 333 = 666$.

Now suppose n is *odd*. We then write $n = 2k + 1$ for a certain integer k. We then get: $n^2 = 4k^2 + 4k + 1$ and $2n + 1 = 4k + 3$. If we divide $4k^2 + 4k + 1$ by $4k + 3$, then the outcome is k and the remainder is $4k^2 + 4k + 1 - k(4k + 3) = k + 1$. Because the remainder must be 1000, we find $k + 1 = 1000$ and therefore $k = 999$. This yields the solution $n = 2 \cdot 999 + 1 = 1999$.

5. $\frac{2}{3}$ The upper half of the square is depicted in the figure. Triangles PST and PSQ are each others images when reflecting in the line PC. Each of them has an area equal to an eighth of the octagon. Triangles QRS and PSQ have equal area, because both triangles have $\frac{1}{2}$ as base length and an equal height. The three triangles together form the grey triangle in the figure, which has base length 1 and height $\frac{1}{2}$. Hence, the grey triangle's area equals $\frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$ and each of the three smaller triangles's area equals $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$. Hence, the area of the octagon is $8 \cdot \frac{1}{12} = \frac{8}{12} = \frac{2}{3}$.



C-problems

C1. (a) An example of a correct sequence is $5, 7, 6, 3, 1, 2$. This sequence consists of six distinct numbers and is 2-composite since $5 + 6 + 1 = 7 + 3 + 2$. It is also 3-composite since $5 + 3 = 7 + 1 = 6 + 2$.

This is just one example out of many possible correct solutions. Below we describe how we found this solution.

We are looking for a sequence $a_1, a_2, a_3, a_4, a_5, a_6$ *that is* 2*-composite and* 3*-composite. Hence, we need that*

$$a_1 + a_4 = a_2 + a_5 = a_3 + a_6 \qquad \text{and} \qquad a_1 + a_3 + a_5 = a_2 + a_4 + a_6.$$

If we choose $a_4 = -a_1$, $a_5 = -a_2$, *and* $a_6 = -a_3$, *then the first two equations hold. The third equation gives us* $a_1 + a_3 - a_2 = a_2 - a_1 - a_3$, *and therefore* $a_1 + a_3 = a_2$. *We choose* $a_1 = 1$, $a_3 = 2$ *(and therefore* $a_2 = 3$*). We obtain the sequence* $1, 3, 2, -1, -3, -2$ *consisting of six distinct integers. If we wish to do so, we can increase all six numbers by* 4 *to get a solution with only positive numbers:* $5, 7, 6, 3, 1, 2$.

(b) A possible solution is $8, 17, 26, 27, 19, 10, 1$. This sequence consists of seven distinct integers and is 2-composite since $8 + 26 + 19 + 1 = 17 + 27 + 10$. It is 3-composite since $8 + 27 + 1 = 17 + 19 = 26 + 10$. It is also 4-composite since $8 + 19 = 17 + 10 = 26 + 1 = 27$.

This is just one example out of many possible correct solutions. Below we describe how we found this solution.

We are looking for a sequence $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ *that is* 2*-,* 3*-, and* 4*-composite. Hence, we need that*

$$a_1 + a_3 + a_5 + a_7 = a_2 + a_4 + a_6, \tag{1}$$
$$a_1 + a_4 + a_7 = a_2 + a_5 = a_3 + a_6, \tag{2}$$
$$a_1 + a_5 = a_2 + a_6 = a_3 + a_7 = a_4. \tag{3}$$

We notice that in equation (1) we have $a_1 + a_5$ *and* $a_3 + a_7$ *on the left, and* $a_2 + a_6$ *and* a_4 *on the right. If the sequence is* 4*-composite, these four numbers are equal. Hence, we find that a* 4*-composite sequence is automatically* 2*-composite as well.*

From the equations in (3) it follows that

$$a_1 = a_4 - a_5, \qquad a_2 = a_4 - a_6, \qquad a_3 = a_4 - a_7.$$

Substituting this in the equations (2), we obtain

$$2a_4 + a_7 - a_5 = a_4 + a_5 - a_6 = a_4 + a_6 - a_7.$$

Subtracting a_4 from each part, we get

$$a_4 + a_7 - a_5 = a_5 - a_6 = a_6 - a_7.$$

Hence, we obtain

$$\begin{aligned} a_4 &= (a_5 - a_6) - (a_7 - a_5) = 2a_5 - a_6 - a_7, \\ a_5 &= (a_6 - a_7) + a_6 = 2a_6 - a_7. \end{aligned}$$

We have thus expressed a_1, a_2, a_3, a_4, and a_5 in terms of a_6 and a_7. Every solution is obtained by a suitable choice of a_6 and a_7 for which the seven numbers become distinct. We try $a_6 = 10$ and $a_7 = 1$, and find:

$$a_5 = 2 \cdot 10 - 1 = 19, \qquad a_4 = 2 \cdot 19 - 10 - 1 = 27, \qquad a_3 = 27 - 1 = 26,$$

$$a_2 = 27 - 10 = 17, \qquad \textit{and} \qquad a_1 = 27 - 19 = 8.$$

Hence, we have found a solution.

(c) The largest k for which a k-composite sequence of 99 distinct integers exists, is $k = 50$. An example of such a sequence is

$$1, 2, \ldots, 48, 49, 100, 99, 98, \ldots, 52, 51.$$

The 99 integers in the sequence are indeed distinct and we see that $1 + 99 = 2 + 98 = \ldots = 48 + 52 = 49 + 51 = 100$, so this sequence is 50-composite.

Now suppose that $k > 50$ and that we have a k-composite sequence $a_1, a_2, \ldots, a_{99}$. Consider the group that contains the number a_{49}. Since $49 - k < 0$ and $49 + k > 99$, this group cannot contain any other number beside a_{49}. Next, consider the group containing the number a_{50}. Since $50 - k < 0$ and $50 + k > 99$, this group cannot contain any other number beside a_{50}. Hence, the numbers a_{49} and a_{50} each form a group by themselves and must therefore have the same value. But this is not allowed since the 99 numbers in the sequence had to be distinct.

C2. (a) The years 2103 to 2109 are seven consecutive interesting years. If there is an earlier sequence of seven, then it must start before 2100. We shall now prove that this is not possible.

Because an interesting year cannot end with the digits 99, the first two digits are the same for all years in a sequence of consecutive interesting years (of four digits). Now suppose we have seven consecutive years starting with digits 20. The seven final digits are consecutive and unequal to 0 and 2, and therefore also unequal to 1. The seven final digits must be the digits 3 to 9, in this exact order. Hence, the third digit must be the only remaining digit, namely digit 1. We conclude that 2013 to 2019 is the only sequence of seven consecutive interesting years between 2000 and 2100.

(b) Suppose that there is a sequence of eight consecutive interesting years between 1000 and 9999. Because an interesting year cannot end with 99, all eight years have the same first two digits. If also the third digit does not change, then there are only 7 possibilities for the last digit, which is not enough. Therefore, there are two consecutive years in our sequence of the shape $abc9$ and $abd0$ with $d = c + 1$. Because there are eight possible final digits, these must be the eight digits unequal to a and b. Hence, both c and $d = c + 1$ must occur as final digit. Because the numbers $abcc$ and $abdd$ cannot occur, this means that in our sequence both $abcd$ and $abdc$ must occur. The difference between these two numbers is 9, and our sequence consists of eight consecutive numbers. This is also not possible. We have obtained a contradiction, and conclude that the assumption that there exists a sequence of eight consecutive interesting years between 1000 and 9999 is wrong.