

First round

Dutch Mathematical Olympiad

20 January – 31 January 2025

Solutions

- A1.** A) At least one of gnomes A and B is wearing a blue hat. If a gnome sees two other gnomes each wearing a blue hat, then the gnome knows he must be wearing a red hat himself, because there are only two blue hats. If a gnome does not know which hat he himself is wearing by looking at two other gnomes, then those two hats are therefore not both blue. In this way, we know that C and D do not both have blue hats, that C and E do not both have blue hats, and that D and E do not both have blue hats. Thus, of the gnomes C, D and E, at most one has a blue hat. This means that out of A and B, there is at least one gnome with a blue hat.
- A2.** B) 3 Denote the age of Alexa by A , the age of Bente by B , and the age of Charissa by C . Then we have $0 \leq A < B < C \leq 9$ and father's age is \overline{AB} (which denotes the number with the digits A and B), mother's age is \overline{AC} , and grandfather's age is \overline{BC} . The problem statement now implies that $\overline{AB} + 32 = \overline{BC}$ and $\overline{AB} + \overline{AC} = \overline{BC} + B$.
- By considering the ones in the first equation, we see that $B + 2 = C$ or $B + 2 = 10 + C$. But in the latter case $B = 8 + C > C$ would hold; this is a contradiction. Hence, $B + 2 = C$, which also immediately yields that $A + 3 = B$ by looking at the tens.
- The second equation boils down to $(10A + B) + (10A + C) = (10B + C) + B$, or $20A + B + C = 10B + B + C$, which yields $B = 2A$. Together with $B = A + 3$ this leads to $2A = A + 3$ and hence $A = 3$ (and $B = 6$ and $C = 8$). So Alexa is 3 years old.
- A3.** B) $\frac{1}{3}$ Denote the amount of flour Laura uses for the muffins by m and the amount of flour Laura uses for the chocolate cake by c . So it holds that $m + c = T$, where T is the total amount of flour (or sugar), because Laura does not bake anything else. Furthermore, Laura uses $\frac{1}{2}m$ sugar for the muffins and $\frac{5}{4}c$ sugar for the chocolate cake. Since she uses as much flour as sugar, $m + c = \frac{1}{2}m + \frac{5}{4}c$. We can rewrite this to $\frac{1}{2}m = \frac{1}{4}c$ and thus $c = 2m$. Combined with $m + c = T$, we find that $m = \frac{1}{3}T$. So, Laura uses $\frac{1}{3}$ of the total amount of flour for the muffins
- A4.** D) $2022 \cdot 2023 - 1$ We can find this solution by first considering small numbers. For example, note that $1 \cdot 2 \cdot 3 \cdot 4 + 1 = 25 = 5^2 = (2 \cdot 3 - 1)^2$ and $2 \cdot 3 \cdot 4 \cdot 5 + 1 = 121 = 11^2 = (3 \cdot 4 - 1)^2$; we can prove that this pattern continues. To do so, first note that

$$\begin{aligned} 2021 \cdot 2024 &= (2022 - 1) \cdot (2023 + 1) \\ &= 2022 \cdot 2023 - 2023 + 2022 - 1 \\ &= 2022 \cdot 2023 - 2. \end{aligned}$$

Denote $2022 \cdot 2023$ with N . Then

$$\begin{aligned} 2021 \cdot 2022 \cdot 2023 \cdot 2024 + 1 &= N(N - 2) + 1 \\ &= N^2 - 2N + 1 \\ &= (N - 1)^2 \\ &= (2022 \cdot 2023 - 1)^2 \end{aligned}$$

Remark: A fast way to see that the other four expressions are incorrect is by looking at the last digit of the given expression, which is a 5. The five answer options end with a 2, 3, 7, 5, and 1, respectively, and their squares then end with a 4, 9, 9, 5, and 1. Only the square of $2022 \cdot 2023 - 1$ ends with a 5.

- A5.** **B) 7** We can get to point $(-1, -1)$ in two moves: first move four squares to the left and three squares up, and then three to the right and four down. Do these two moves three times, then you are at the point $(-3, -3)$. From there, you can get to the point $(1, 0)$ in one move: four to the right and three up.

We show that less than seven moves is not possible. From $(0, 0)$ to $(1, 0)$ you have to go up as much as down. This can be done by making for each move with 3 up also a move with 3 down and for every move with 4 up also a move with 4 down. Another possibility is to move four times 3 up and three times 4 down, or vice versa. In the latter case, you already need seven moves (this also happens in the solution above). So if you want to use less than seven moves, you have to assume the first case and then the moves come in pairs with equal amounts of movement up and down. If you go up 4 and down 4, then you move horizontally twice a distance of 3. In total you move an even number of squares (0 or 6). If you move 3 up and 3 down, you move 0 or 8 squares horizontally. So you always move an even number of squares horizontally and can never reach $(1, 0)$.

So in total, you need at least seven moves.

- A6.** **D) $18\frac{3}{4}$** Call the two sides of the triangle that are not the hypotenuse a and b . We are looking for the area of the triangle, which is $\frac{1}{2}ab$. The Pythagorean theorem now gives that $a^2 + b^2 = 11^2$. We also know that $a + b = 25 - 11 = 14$. Then

$$14^2 = (a + b)^2 = a^2 + b^2 + 2ab = 11^2 + 2ab$$

and it follows that $2ab = 196 - 121 = 75$. So $\frac{1}{2}ab = \frac{75}{4} = 18\frac{3}{4}$.

- A7.** **C) 178°** First we consider the triangle EAL . Since both polygons are regular, $|AE| = |AB| = |AL|$ and so EAL is an isosceles triangle. The angle at A is equal to the angle of a regular nonagon minus the angle of a regular pentagon. Since you can divide an n -gon (for $n \geq 3$) into $n - 2$ triangles of 180° each, the angle of a regular n -gon is equal to $\frac{n-2}{n} \cdot 180^\circ$. So for a pentagon, that angle is 108 degrees and for a nonagon, 140 degrees. So the angle at A in the triangle EAL is $140 - 108 = 32$ degrees. This is the apex of the isosceles triangle: so the other two angles are $\frac{1}{2}(180 - 32) = 74$ degrees.

Now we can calculate the required angle: it is 360° minus an angle of the pentagon minus the angle we just calculated, so $360^\circ - 108^\circ - 74^\circ = 178^\circ$.

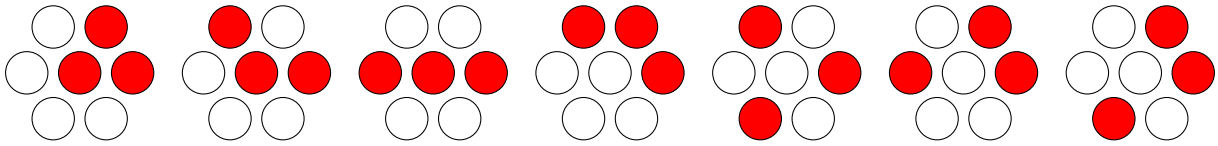
- A8.** **D) 1025** When Babette calculates the distance between two crosses, she subtracts the smallest value at a cross from the largest value at a cross. Then she adds everything together. Suppose Babette puts her crosses at the numbers $a_1, a_2, \dots, a_{1001}$ with $a_1 < a_2 < \dots < a_{1001}$. Now for the cross at a_1 , it always appears in the outcome with a minus sign: the distances to the other crosses are $a_2 - a_1, a_3 - a_1, \dots, a_{1001} - a_1$. This contributes $-1000 \cdot a_1$ to the total outcome. Now looking at the cross at a_2 , this number appears once with a plus and 999 times with a minus in the total sum: with a plus in $a_2 - a_1$ and with a minus in $a_3 - a_2, a_4 - a_2, \dots, a_{1001} - a_2$. In total, this adds $-998 \cdot a_2$ to the outcome. Continuing in this way, we see that the total sum of all distances is given by

$$-1000 \cdot a_1 - 998 \cdot a_2 - 996 \cdot a_3 - \dots - 2 \cdot a_{500} + 0 \cdot a_{501} + 2 \cdot a_{502} + \dots + 1000 \cdot a_{1001}.$$

Clearly, this outcome is as large as possible if the first 500 a_i s are as small as possible and the last 500 a_i s are as large as possible. The number a_{501} , which does not contribute to the outcome, is the only one for which there are multiple spots to choose from: the 1025 spots in the middle.

- B1.** **7** By systematic counting we will show that there are seven possibilities for the bouquet. If the center rose is red, then there are 2 red roses in the ring around it. There can be

0, 1 or 2 white roses between them; so these are 3 possibilities. If the center rose is white, then all three red roses are in the ring around it. Either all three are directly next to each other, or all three are not next to each other (i.e., in the ring the white and red roses are alternating), or there are exactly two of them next to each other; in this last case, the third rose can be in two places compared to the first two. In total, we get $3 + 4 = 7$ bouquets.

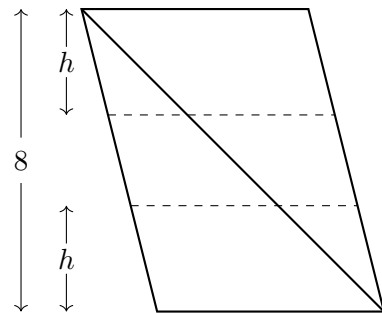


- B2.** $23 : 57 : 33$ Notice that the hours always start with 0, 1 or 2, and the minutes and seconds with 0, 1, 2, 3, 4 or 5. That's already fifteen places where digits 0 through 5 should be, of which we have a total of eighteen available. In addition, if we start an hour with a 2, the next digit must also come from 0 to 3. We do want to do that at least in the last hour, otherwise our maximum time difference will be less than 20 hours. The two remaining digits from 0 to 5 are best used in the earliest time. It will then look like this:

$$\begin{aligned} 00 & : 02 : 2X \\ 1X & : 3X : 3X \\ 1X & : 4X : 4X \\ 1X & : 4X : 5X \\ 23 & : 5X : 5X \end{aligned}$$

Note that if we use a 1 in the earliest time instead of a 2, then any of the middle times must begin with a 2 and therefore needs an additional digit from 0 to 3. Then the earliest time becomes at least $00 : 06$ and that is later than how it appears above. The earliest possible time is thus $00 : 02 : 26$ and the latest possible time is $23 : 59 : 59$. (In the middle three times, the exact placement of the 3, 4, 5 does not matter). However we further distribute the digits 6 through 9 among the X s, they always give valid times. So the maximum time difference is $23 : 59 : 59 - 00 : 02 : 26 = 23 : 57 : 33$.

- B3.** $3\frac{1}{5}$ We complete the triangle with a congruent triangle to form a parallelogram, as shown on the right. Just as the middle part of the triangle is one-fifth of the total triangle, we see that the middle part of the parallelogram is one-fifth of the total parallelogram. Therefore, since the area of a parallelogram is equal to base times height, and the three pieces of the parallelogram are also parallelograms with all the same base, the middle height is equal to one-fifth of the total height. In other words, $8 - 2h = \frac{8}{5}$, and solving gives that $h = \frac{16}{5} = 3\frac{1}{5}$.



- B4.** $4, 5, 8, 6$ or $5, 4, 8, 6$ or $5, 8, 4, 9$ or $6, 9, 8, 4$ or $7, 8, 9, 6$ or $8, 5, 4, 9$ or $8, 7, 9, 6$ or $9, 6, 8, 4$
 When Lydia starts with three numbers, she writes down two more numbers after that. Those two numbers each have at most two digits, so then there are a total of seven digits on the board. Those are too few to contain all the digits from 1 to 9. Lydia therefore needs (at least) four numbers. If she uses four numbers, she writes down three more numbers after that, so at most ten digits in total. We can find, with a bit of trying, a solution with four numbers: there are eight possibilities in total, see above.