First round
Dutch Mathematical Olympiad

22 January – 2 February 2024

Solutions

A1. D) 5 The squares of two digits are 16, 25, 36, 49, 64, and 81. We start with each of these squares, and see which digit we can stick behind it such that the last two digits also form a square again. For example: after 16, a 4 is possible because of 64 (and no other digit is possible after a 6), then comes a 9 to form the square 49, and then it ends because no square starts with a 9. With the six different starting squares, you can make at most the following numbers:

- 1649
- 25
- 3649
- 49
- 649
- 81649

The largest number in this list, 81649, consists of five digits.

A2. E) 25 We make a table of how many days it does or does not rain, and how many days rain is or is not predicted and so Caitlin does or does not bring an umbrella.

		Does it rain? yes	no
Rain predicted?	yes	a	b
	no	0	c

It never happens that it rains and Caitlin doesn't have an umbrella with her, which is why there is a 0 at the bottom left. For the other three numbers, it holds that $a+b+c=31$ (the number of days in October), $a+b=16$ (the number of days rain was predicted) and $a+c=21$ (the number of days the prediction was correct). We are looking for $b+c$, the number of days it did not rain. From $a+b+c=31$ and $a+b=16$, it follows that $c=15$. Combined with $a+c=21$, this means that $a=6$. Finally, then $b+c=31-6=25$.

Alternative solution. On the $31-16=15$ days in October when no rain was expected, the forecast was apparently correct. So of the other 16 days (on which rain was expected), the forecast was correct only on $21-15=6$ days. So it rained on exactly 6 days, so on $31-6=25$ days it did not.

A3. C) $\frac{2}{3}$ We calculate the following numbers in this sequence:

- 6;
- 15;
- $\frac{15}{6}\cdot 2=5$;
- $\frac{5}{15}\cdot 2=\frac{2}{3}$;
- $\frac{2}{3}:5\cdot 2=\frac{2}{3}\cdot\frac{1}{5}\cdot 2=\frac{4}{15}$;
- $\frac{4}{15}:\frac{2}{3}\cdot 2=\frac{4}{15}\cdot\frac{3}{2}\cdot 2=\frac{4}{5}$;
- $\frac{4}{5}:\frac{4}{15}\cdot 2=\frac{4}{5}\cdot\frac{15}{4}\cdot 2=6$;
- $6:\frac{4}{5}\cdot 2=6\cdot\frac{5}{4}\cdot 2=15$;
- ...

We see that the sequence is periodic: the same six numbers repeat over and over again. The first number is a 6, the seventh number is a 6, ..., the 97th number is a 6. So the one hundredth number is $\frac{2}{3}$.

A4. E) 5 We make the boxes of the 3×3 grid alternately grey and white and give each box a letter, see the figure on the right. A snake always swings alternately between the white and grey boxes, and with it the numbers are alternating even and odd. Since there are five odd numbers and five white boxes, the odd numbers come on the white boxes, in places a, c, e, g, and i. We calculate the total score of this grid as follows:

a	b	c
d	e	f
g	h	i

$$\begin{aligned}(b+d)+(a+e+c)+(b+f)+(a+e+g)+(b+d+f+h)\\ +(c+e+i)+(d+h)+(e+g+i)+(f+h)\\ =2(a+b+c+d+e+f+g+h+i)+(b+d+f+h)+2e\end{aligned}$$

This is equal to twice the sum of all numbers, plus the sum of all even numbers, plus twice e. The first two terms are the same for every snake. So the total score of the grid depends only on the number that will appear in place e. That is an odd number, as we saw, and there are five possibilities for that. Below we see examples showing that any odd number can actually end up in the middle.

5	4	3
6	1	2
7	8	9

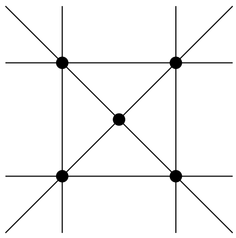
1	2	9
4	3	8
5	6	7

1	2	3
6	5	4
7	8	9

1	2	3
8	7	4
9	6	5

1	2	3
8	9	4
7	6	5

A5. D) 6 We can get 6 lines as follows. We take the extended sides of a square together with the two diagonals. The 5 points are now the four vertices of the square and the centre, see also the figure on the right.



We will show that 7 (or more) lines are not possible. We do this by contradiction: assume a situation with 7 lines does exist. In general, 7 lines intersect in $\frac{7 \cdot 6}{2} = 21$ points, but there are two reasons why there could be fewer.

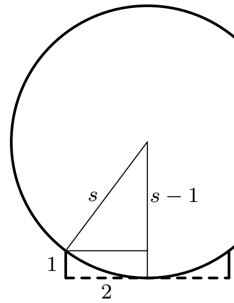
First, lines can be parallel, so they do not intersect. Since each line must contain at least 2 of the 5 points, it is not possible for 3 lines to be parallel (because that would require $3 \cdot 2 = 6$ points). Thus, of the 7 lines, there are at most 3 pairs that are parallel, and thus at most 3 intersection points less due to parallelism. This leaves at least $21 - 3 = 18$ intersection points.

Second, intersections can coincide. This happens when there are more than two lines passing through one point. If 3 lines pass through one point, 3 intersection points coincide. If there are 4 lines passing through a point, 6 intersection points coincide, and so on. The at least 18 remaining intersection points must coincide in the 5 points we started with. That means there is a point, say P, in which at least 4 intersection points coincide (because $5 \cdot 3 = 15 < 18$). So through that point there are at least 4 different lines, say ℓ_1, ℓ_2, ℓ_3 and ℓ_4.

On each of those 4 lines must lie another of the 5 points, say points P_1, P_2, P_3 and P_4. These points cannot coincide. Since we have now already found all 5 points, there cannot be a fifth line that goes through point P. Each of the remaining 3 lines intersects at least 3 of the lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 (because it is parallel to at most one of those lines). This means that the remaining lines must contain at least 3 of the points P_1, P_2, P_3 and P_4. However, then it follows that the 3 remaining lines are all the same line, which is a contradiction.

We conclude that 7 (or more) lines are not possible and that 6 is the maximum number of lines.

A6. B) $2\frac{1}{2}$ We draw a line segment from the centre of the ball to the edge of the holder, and from the centre of the ball to the table. We also draw a line parallel to the table. We are looking for s, the radius of the ball.



We see a right-angled triangle with rectangular sides 2 and $s-1$ and hypotenuse s. The Pythagorean theorem gives that $2^2+(s-1)^2=s^2$. Expanding this gives $4+s^2-2s+1=s^2$, or $2s=5$, so $s=\frac{5}{2}=2\frac{1}{2}$.

A7. C) There do not exist $a>0$ and $b>0$ with $a+b<a\cdot b<\frac{a}{b}$. We will show that there are no $a>0$ and $b>0$ such that $a+b<a\cdot b<\frac{a}{b}$. It follows from the second inequality that $a\cdot b^2<a$, or $b^2<1$. It follows that $b<1$. But then we see that $a\cdot b<a\cdot 1=a$, while $a+b>a+0=a$, which contradicts $a+b<a\cdot b$.
For completeness, we also show that, for the other options A), B), and D), there do exist $a>0$ and $b>0$ with the desired property. For A) you can take $a=\frac{2}{3}$ and $b=\frac{2}{3}$; then you get $\frac{4}{9}<1<\frac{4}{3}$. For B) you can take $a=1$ and $b=\frac{1}{2}$; then you get $\frac{1}{2}<\frac{3}{2}<2$. Finally, for D) you can take $a=\frac{1}{2}$ and $b=2$; then you get $\frac{1}{4}<1<\frac{5}{2}$.

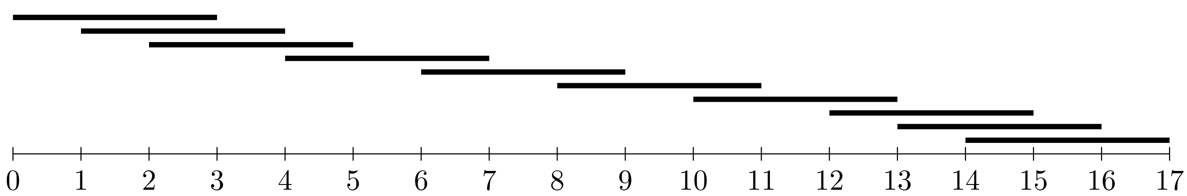
There are also other pairs (a,b) that you can use to exclude options A), B) and D).

A8. D) 17 Note first of all that something must happen at the beginning of each hour; either a candle must be lit or a candle must go out (or both). Since there are 10 candles, there are 20 such events. So this process can take at most 19 hours. However, it turns out that in the first 5 hours (i.e. in the first six events) there must be a moment in which there are two events at the same time. We prove this by contradiction.

Suppose exactly one event occurs every hour. At $t=0$, Birgit lights a candle. At $t=1$ and $t=2$, she has no choice but to light a second and third candle. At $t=3$, the first candle goes out, so Birgit does nothing else. At $t=4$ and $t=5$, the second and third candle go out, so Birgit still does not light another candle. Between $t=5$ and $t=6$, there is now no candle burning, but this contradicts the requirements given in the problem statement.

The same argument can be used to show that in the last 5 hours, there must be a moment with two events at the same time. So that means there are two moments with two events at the same time. So the process can take at most 17 hours.

The schematic below shows that it is indeed possible in 17 hours. Every horizontal bar represents the three burning hours of a candle. Note that only one event occurs every hour, apart from $t=4$ and $t=13$ when both a candle goes out and one is lit.



B1. 400 Denote the number of kilometres of the entire route by r. Then the distance from the Janssen family's home to the stopover equals $\frac{1}{2}r$. This is also equal to 150 kilometres plus one-fifth of the distance from the border to the final destination, which is $150+\frac{1}{5}(r-150)$. We find that $\frac{1}{2}r=150+\frac{1}{5}(r-150)$. Solving this equation gives $r=400$.

B2. 21 There are 16 students and $2^4=16$ possible ways of answering the questions. Since the number of students equals the number of possible ways of answering and everyone has a different number of points, each combination of questions must be worth a different number of points. In particular, each question is worth a different number of points. We will now systematically try how many points the lowest scoring question could be worth.

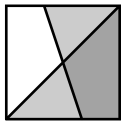
Suppose this question is worth 1 point. Then the question you get the most points for is worth 5 points. There are then at least 0 and at most $1+3+4+5=13$ points to be earned for the whole test, so at most 14 possible scores. But we need at least 16 possible scores, so this cannot be the case.

Suppose the question for which you get the least points is worth 2 points. Then the question for which you get the most points is worth 6 points. There are three possible options for the number of points per question: $2,3,4,6$ or $2,3,5,6$ or $2,4,5,6$. The second option is impossible, because the student who got the questions right for 2 and 3 points has as many points as the student who only got the question right for 5 points. For a similar reason, the first and last options are impossible: then the student who got the questions for 2 and 4 points right would have as many points as the student who only got the question for 6 points right.

We therefore look at what happens if the question for which you get the fewest points is worth 3 points, and the question for which you get the most points is worth 7 points. There cannot be a question worth 4 points, because then the student who got the questions for 3 and 4 points right would have as many points as the student who only got the question for 7 points right. So the four questions are worth 3, 5, 6 and 7 points and we can check that all the possible combinations of zero, one, two, three, and four questions right, give a different number of points. So Floor had at least $3+5+6+7=21$ points on the test.

B3. $\frac{1}{6}$ The dark grey area, plus the two small light grey triangles on the bottom left and right, together form a triangle that is exactly $\frac{1}{4}$ part of the square. So to determine the area of the dark grey part, we need the area of these small light grey triangles. The base of such a triangle is $\frac{1}{3}$.

For the height, we look at the lower left quarter of the picture, see the figure on the right. There you see two congruent light-grey triangles: their height is therefore half the height of the figure, which is a quarter of the original square.



Thus, the area of a light grey triangle is $\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{24}$. It follows that the area of the dark grey region is equal to $\frac{1}{4}-2\cdot\frac{1}{24}=\frac{1}{6}$.

B4. 10 Suppose v litres of water per minute flow through one tap and the bath empties at w litres per minute when the taps are off. Assume the bath can hold b litres of water. We want to know how long it takes until a full bath empties, which is $\frac{b}{w}$.

On Monday, the bath first fills at v litres per minute, and then empties at $w-v$ litres per minute. Together, this takes $\frac{b}{v}+\frac{b}{w-v}$ minutes. On Tuesday, the bath first fills up at $2v$ litres per minute, then empties at $w-2v$ litres per minute. This takes the same amount of time as on Monday. Together, this gives the equation

$$\frac{b}{v}+\frac{b}{w-v}=\frac{b}{2v}+\frac{b}{w-2v}.$$

Equating the fractions, cross-multiplying and simplifying yields that $w=3v$. (You can easily check this by entering it into the equation above.) In other words, draining the bathtub is three times faster than filling it up with one tap.

Finally, we make use of the fact that Pjotr spent 45 minutes both days. Substituting $v=\frac{1}{3}w$ into the formula for Monday, we find

$$45=\frac{b}{v}+\frac{b}{w-v}=\frac{b}{\frac{1}{3}w}+\frac{b}{\frac{2}{3}w}=\frac{9}{2}\cdot\frac{b}{w}.$$

It follows that $\frac{b}{w}=10$.