First round
Dutch Mathematical Olympiad

17 January – 27 January 2022

Solutions

A1. C) 12 The difference between the areas of the large island and the medium island is 1 km^2 more than the area of the small island. In other words, the area of the medium island plus the area of the small island is 1 km^2 less than the area of the large island. Hence, the sum of all three areas of the islands is 1 km^2 less than two times the area of the large island. Therefore, twice the area of the large island equals $23 + 1 = 24$, and the large island has an area of 12 km^2.

A2. C) 14 If Kevin draws n lines, then the angle of 360° at vertex P is divided into $2n$ pieces. If Kevin would draw 13 lines with equal angles between adjacent lines, then that angle would be exactly $\frac{1}{26} \cdot 360° = 13\frac{11}{13}°$. We conclude that 13 lines are not sufficient. Would Kevin draw 14 lines with an angle of at least 13° between adjacent lines, then the sum of the 28 angles is at least $28 \cdot 13° = 364°$, which is impossible. Hence, using 14 lines, you can be sure that there is an angle of less than 13°.

A3. C) 70 In six consecutive years the age of grandma is an integral multiple of the age of her granddaughter Sofie. In the first year of these six consecutive years we call grandma's age g and Sofie's age s. Then g is divisible by s, but also $g - s$ is then divisible by s. In the next year the age of grandma is $g + 1$ and that of Sofie $s + 1$. Then $g + 1$ is divisible by $s + 1$, but $(g + 1) - (s + 1) = g - s$ is also divisible by $s + 1$. A year later the age of grandma is $g + 2$ and that of Sofie is $s + 2$. Then $g + 2$ is divisible by $s + 2$ and also $(g + 2) - (s + 2) = g - s$ is divisible by $s + 2$.

Continuing for another three years, we find that $g - s$ is divisible by the consecutive numbers s, $s + 1$, $s + 2$, $s + 3$, $s + 4$ and $s + 5$. If these contain 5, 6 and 7, then $g - s$ has to be at least $5 \cdot 6 \cdot 7 = 210$ so grandma has to be at least 210 years old. That is impossible. If s is at least 6 (so 5 is no longer one of the numbers) and at most 9, then the numbers contain 9, 10 and 11 and grandma is more that 990 years old. If s is at least 10 and at most 13, then the numbers contain 13, 14 and 15 and grandma has to be even older. Et cetera. The only realistic option is now to take $s = 1$, so we have the numbers 1 to 6. (Note that $s = 0$ is not possible, because then grandma and Sofie would both be 0 years old.). Now we need $g - s$ to be a multiple of $5 \cdot 6 = 30$. That is not divisible by 4, so $g - s$ even has to be a multiple of 60. Indeed, 60 is divisible by the numbers 1 to 6. (We call this the *least common multiple* of these numbers.) We conclude that $g - s = 60$, because if $g - s = 120$ or an even bigger multiple of 60, grandma's age is not humanly possible.

Since $g - s = 60$ and $s = 1$, grandma is 61 years old in the first year and Sofie is 1 year old. Indeed, 61 is a multiple of 1. Going on we see that 62 is a multiple of 2, 63 is a multiple of 3, 64 is a multiple of 4, 65 is a multiple of 5 and 66 is a multiple of 6. In the seventh year we get 67, which is not a multiple of 7. In the two following years we get 68 and 69 that are not multiples of 8 and 9, respectively. But if grandma is 70 years old, Sofie is 10, and their ages are again a multiple of each other.

A4. E) 56 There are not so many possibilities for the four digits of the number. We list them all.

One 6 and three 0s. A four digit number cannot start with a 0, hence there is only 1 possibility.

One 5, one 1, and two 0s. We can start with the 1, then there are 3 possibilities for the position of the 5. We could also start with the 5 and then there are 3 possibilities for the position of the 1. Altogether there are $3 + 3 = 6$ possibilities in this case.

One 4, one 2, and two 0s. Using exactly the same argumentation as in the previous case, we find 6 possibilities.

One 4, two 1s, and one 0. For the 0 we choose a position not at the start; this can be done in 3 ways. For the 4 there are 3 possibilities left, and the other two digits are 1s. Hence, there are $3\times3 = 9$ possibilities in total.

Two 3s and two 0s. We have to start with a 3. For the other 3 we can chose out of three positions; this can be done in 3 ways.

One 3, one 2, one 1, and one 0. There are 3 possibilities for the 0, then 3 remaining possibilities for the 1, then 2 possibilities for the 2, and the last remaining digit must be the 3. These are $3\times3\times2 = 18$ possibilities.

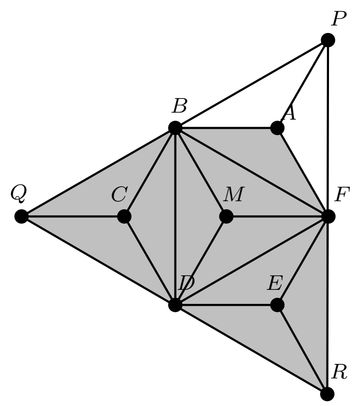
One 3 and three 1s. The 3 can be put in each of the four possible positions; the remaining digits are 1s. This can be done in 4 ways.

Three 2s and one 0. The 0 has to be put in one of the last three positions; this can be done in 3 ways.

Two 2s and two 1s. For the first 1 we can chose one of the 4 possible positions and for the second 1 one of the remaining 3 positions. But be aware: now we counted each possibility for the two 1s twice (because it does not matter which 1 is the first). Hence, there are $\frac{1}{2}\times4\times3 = 6$ possibilities.

In total, we found $1 + 6 + 6 + 9 + 3 + 18 + 4 + 3 + 6 = 56$ solutions.

A5. C) $\frac{6}{5}$ We draw the centre M of the triangle, which is also the centre of the hexagon. Draw line segments from M to the vertices B, F, and D of the hexagon. Also draw line segments AP, CQ, and ER. Because the triangle is equilateral and the hexagon is regular, each of the six line segments is part of a symmetry axis for the triangle, which is also a symmetry axis for the hexagon. Then we draw triangle BFD. Now we divided the big triangle into twelve small triangles, as you can see in the figure.



Because of the symmetry in the figure, these small triangles are congruent: they are isosceles triangles with angles 30°, 30°, and 120°, and the length of their base is half of the length of one side of the big triangle. The grey pentagon has an area consisting of 10 small triangles, and the big triangle has an area consisting of 12 small triangles. Therefore, the area of the big triangle is $\frac{12}{10} = \frac{6}{5}$.

A6. D) 70 Suppose there are R red balls, B blue balls, and W white balls in the box. Using the data in the problem statement, we get that

$$W + B = 4R \quad \text{and} \quad R + B = 6W.$$

There are many ways to solve this system of equations: we give one solution. Subtracting one equation from the other yields $W - R = 4R - 6W$, or $7W = 5R$. Because 7 is a prime number and 5 is not divisible by 7, the number R must be divisible by 7. Moreover, R also has to be even, hence R is a multiple of 14. If we take $R = 14$, then we get $W = \frac{5}{7}R = 10$ and $B = 4R - W = 46$. If we take R greater than 14, then R is at least 28 and we get that W is at least 20 and B is at least 92. The total number of balls is smaller than 100, hence this is impossible and R must be 14. In total, there are $14 + 10 + 46 = 70$ balls in the box.

A7. B) C won against B Team A won in the first and third round, and therefore lost in the second round. Team C won in the first round and hence did not play against team A in that round. Team D lost in the second round and hence did not play against team A in that round. Team D also lost in the first round, because both A and C won in that round, hence team D must have won in the last round. Team A also won in that round, hence the game between A and D was in the first round. The other game in the first round was between teams B and C, and C won.

A8. A) 1 On the bottom die we see 3 pips on the front face, hence there are 4 pips on the back face. On the right side of the bottom die can only be a 1, 2, 5, or 6. With a 5 or 6, the total number of pips on the back must be at least 15 or 18, which is impossible. If there is a 1 on the right face, then the top face has 2 pips (see the figure) and the bottom face of the top die must be a 7. This is also impossible. Hence, there is a 2 on the right face of the bottom die. On the bottom face of that same die, there must be 1.

To check everything, we look at the rest of the tower. On the top face of the bottom die, there is a 6, hence the bottom face of the top die contains 3 pips. Then there are four ways left in which we can place the top die. We know that the total number of pips on the back of the tower must be three times the total number of pips on the right side of the tower. Trying all possibilities, we find that the right face of the top die has 1 pip, and the back face has 5 pips. Indeed, we then have $3 \cdot (2+1) = 4+5$.

B1. 17 If we add up two of the numbers 1 up to and including 15, we can never get a square greater than $25 = 5^2$. We can figure out for each number how much we need to add to it to get a square, and in how many ways this can be done. Starting with 5 for example: you can add 4 to get $5+4=9=3^2$, or you can add 11 to get $5+11=16=4^2$. These are all possibilities: to get the next square, you need a number greater than 15. This means that 5 has to be between 4 and 11 in the sequence.
If we do this for all numbers, we see that there are at least two possibilities for almost all numbers to create a square. The exceptions are 8 and 9, for which there is only one way to do this: $8+1=9=3^2$ and $9+7=16=4^2$. Therefore the numbers 8 and 9 must be on the end of the sequence. When we add the first and last number in the sequence, we get $8+9=17$. With a bit of trying, we find that there is only one possibility for the whole sequence:

$$9 \quad 7 \quad 2 \quad 14 \quad 11 \quad 5 \quad 4 \quad 12 \quad 13 \quad 3 \quad 6 \quad 10 \quad 15 \quad 1 \quad 8$$

B2. $4\frac{1}{2}$ The radius of the circle is 3. We can split the small square into two triangles along the diagonal. Using the long side as the base (length 6), these triangles have height 3 and hence area $\frac{1}{2} \cdot 6 \cdot 3 = 9$. Hence, the four small triangles on the outside have a joint area of $36 - 2 \cdot 9 = 18$. Therefore, the grey triangle has area $\frac{18}{4} = 4\frac{1}{2}$.

B3. 117 Let's first look at the special pairs consisting of one mathematician and one biologist that know each other. Let w be the number of mathematicians and let b be the number of biologists. Because each mathematician knows four biologists, the number of special pairs equals $4w$. On the other hand, each biologist knows nine mathematicians, hence the number of special pairs also equals $9b$. The two numbers must equal, therefore $4w = 9b$, or $b = \frac{4}{9}w$.

Moreover, we know that each mathematician knows twice as many people as each biologist. Each mathematician knows all $w-1$ other mathematicians plus four biologists, $w+3$ people in total. Each biologist knows all $b-1$ other biologists plus nine mathematicians, $b+8$ people in total. Now every mathematicians knows twice as many people as each biologists, hence $w+3 = 2\cdot(b+8)$, or $w = 2b+13$. When we substitute $b = \frac{4}{9}w$, we get that $w = 2 \cdot \frac{4}{9}w + 13$ and the solution of this equation is $w = 117$.

B4. 29 There is exactly one beetle that does not crawl anywhere, the beetle on the bottom left. All other beetles must crawl either one square to the left or one square diagonally to the bottom right. At most two beetles can end up on the same square. On the board below, the beetles are distributed in groups in which some of the beetles can end up together on the same square: for example, the beetle on the square on the top left with the N can end up on the same square as the beetle on the square with the letter N in the second row. It would also be possible for beetle N in the second row to end up on the same square as beetle N of the third row.

N	O	P	Q	R	S	T	U
L	M	N	O	P	Q	R	S
J	K	L	M	N	O	P	Q
H	I	J	K	L	M	N	O
F	G	H	I	J	K	L	M
D	E	F	G	H	I	J	K
B	C	D	E	F	G	H	I
A	A	B	C	D	E	F	G

There are four N beetles, which can end up on two squares with two beetles each. For some other groups, there is an odd number of beetles, which forces one of the beetles in this group to end up on a square alone. This holds for the groups T, U (these beetles cannot end up on the same square as any other beetle), D, E, P, Q (here there are three, of which two can end up together and one remains alone).

Therefore, there are at least 6 squares on which just one beetle ends up and at most $\frac{1}{2}(64-6) = 29$ squares with two beetles. For each square with two beetles, there is also an empty square somewhere else on the board, hence there are at most 29 empty squares. From the explanation before it follows that it can indeed happen that 29 squared end up empty.