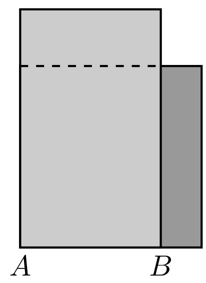
First Round
Dutch Mathematical Olympiad

20 January – 30 January 2020

Solutions

A1. E) $13\frac{1}{3}$ The right edge of the rectangle intersects the top and bottom edge of the square at a point three quarters along those edges. That is, $AB = 7\frac{1}{2}$ cm. The rectangle has the same area as the square, namely $10 \text{ cm} \times 10 \text{ cm} = 100 \text{ cm}^2$. We therefore find that the long edge of the rectangle has length

$$\frac{100 \text{ cm}^2}{7\frac{1}{2} \text{ cm}} = \frac{200}{15} \text{ cm} = \frac{40}{3} \text{ cm} = 13\tfrac{1}{3} \text{ cm}.$$



A2. C) It was Kwak. Exactly one of Kwik's statements on Sunday and Monday must have been true. Suppose that his statement on Sunday was true. In that case, he was lying on Monday and therefore also on Tuesday. If, on the other hand, his statement on Monday was true, then he must have been lying on Sunday and therefore also on Saturday. Hence, the days that Kwik was lying were either Monday and Tuesday, or Saturday and Sunday.

In the same way, we can consider the two statements made by Kwak. We find that Kwak was either lying on Sunday and Monday, or on Tuesday and Wednesday. Since Kwik and Kwak never lie on the same day, we are left with only one possibility: Kwik was lying on Saturday and Sunday, and Kwak was lying on Tuesday and Wednesday.

On Monday, both Kwik and Kwak were telling the truth. This means that Kwek and Kwik are innocent. We conclude that Kwak was the one who ate the sweets.

A3. D) 27 Let a and b be the two digits, where a is the smaller of the two (or a equals b). The number is vain if $a + b \geqslant a \cdot b$. We will consider two cases and determine in each case when the number is vain.

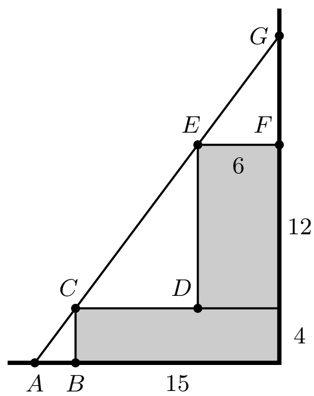
Suppose that $a \geqslant 2$. Then, $a \cdot b \geqslant 2 \cdot b = b + b \geqslant a + b$, which is precisely the reverse inequality needed for being vain. So the number is vain if and only if equality holds. This means that we must have $a = b$ (because we must have $b + b = a + b$) and also $a = 2$ (since $b > 0$ and we must have $a \cdot b = 2 \cdot b$). The only vain number in this case is therefore 22.

Now suppose that $a \leqslant 1$. Then, $a \cdot b \leqslant b \leqslant a + b$. So in this case, the number is always vain. We will count the number of two-digit numbers whose smallest digit equals 0 or 1. These are the numbers 10 to 19 and the numbers $20, 21, 30, 31, \ldots, 90, 91$. Hence, we have 26 vain numbers in this case.

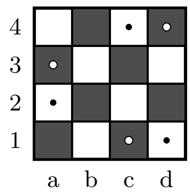
We conclude that there are $1 + 26 = 27$ vain two-digit numbers.

A4. A) 30 Consider the right-angled triangle CDE in the figure. We see that $CD = 15 - 6 = 9$ and $DE = 12$. By the Pythagorean theorem we obtain $CE = \sqrt{9^2 + 12^2} = 15$.

Triangles EFG and CDE are similar, so $EG : EF = CE : CD$. We find that $EG : 6 = 15 : 9$, and therefore $EG = \frac{6 \cdot 15}{9} = 10$. In the same way, we obtain $AC : BC = CE : DE$, and therefore $AC : 4 = 15 : 12$. It follows that $AC = \frac{4 \cdot 15}{12} = 5$. We conclude that the length of the ladder equals $AC + CE + EG = 5 + 15 + 10 = 30$.



A5. C) 10 We colour the squares of the 4×4 board according to a chessboard pattern and denote the squares by a1 to d4 as in the figure. We see that the grass hoppers on the white squares all jump to a black square and that the grass hoppers on a black square all jump to a white square.



Since no two of the squares a3, c1, and d4 are adjacent to the same square, the grass hoppers starting on the squares a3, c1, and d4 will land on three different white squares. In the same way we see that the three grass hoppers starting on a2, c4, and d1 will land on three different black squares. Hence, at least six squares will be occupied after the jumping.

Conversely, we can arrange that after jumping no more than six squares are occupied. Indeed, the grass hoppers starting on a black square can all jump to one of the three squares a2, c4, and d1, whereas the grass hoppers starting on a white square can all jump to one of the three squares a3, c1, and d4.

We conclude that the smallest possible number of occupied squares is 6 and the largest possible number of empty squares is $16 - 6 = 10$.

A6. E) 9 It follows from the first row that

$$\text{ADF} + \text{F} = \text{ABC}.$$

The digit C is therefore even and not equal to 0. This leaves the possibilities 2, 4, 6, and 8. From the middle row we see that

$$\text{C} \cdot \text{GC} = \text{ADD}.$$

Since $2 \cdot 2 = 4$, $4 \cdot 4 = 16$, $6 \cdot 6 = 36$, and $8 \cdot 8 = 64$, it follows that digit D must be 4 or 6. The last row gives

$$\text{D} \cdot \text{GD} = \text{CEF}.$$

Since $4 \cdot 4 = 16$ and $6 \cdot 6 = 36$, the digit F must be 6. Because different letters represent different digits, D must be 4. The first row now gives

$$\text{A46} + 6 = \text{ABC}.$$

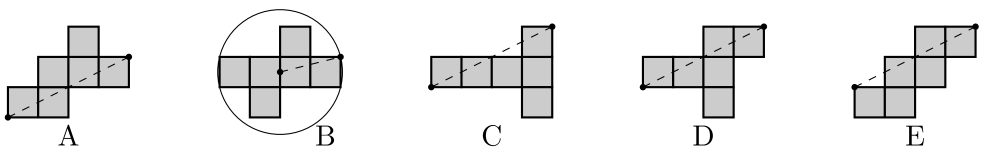
We therefore see that B equals 5 and C equals 2. The first column now reads

$$\text{A52} + \text{A44} = \text{2E6}.$$

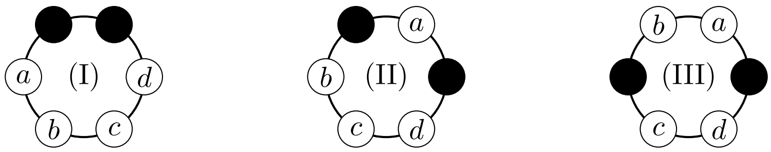
Hence, digit E equals 9. *In addition we also obtain* A = 1 *and* G = 7.

A7. B) B If a figure has radius r, then the distance between any two points of that figure can be at most $2r$. In each of the figures A, C, D, and E we have indicated two points at distance $2\sqrt{5}$. Each of these figures must therefore have a radius of at least $\sqrt{5}$.

The radius of figure B, however, is less than $\sqrt{5}$. As indicated, figure B fits within a circle whose radius squared equals $2^2 + (\frac{1}{2})^2 < 5$. We conclude that figure B has the smallest radius.



A8. B) 11 We split the bracelets into three groups: (I) the black beads are next to each other, (II) there is one bead between the two black beads, and (III) the black beads are in opposite positions. For each of the three groups we count the number of ways to place the remaining beads.



Group (I) We must choose in which two of the positions a, b, c, d we put the white beads. There are six possibilities: ab, ac, ad, bc, bd, and cd. The choices for bd and ac give the same bracelet. The choices for cd and ab also give the same bracelet. In total we therefore find 4 distinct bracelets in group (I).

Group (II) Suppose that a is a white bead. This gives two distinct bracelets: ac white or ab white. Choosing ad to be white gives the same bracelet as choosing ab to be white. If we choose a to be grey instead of white, we also obtain two distinct bracelets. So group (II) also has 4 distinct bracelets.

Group (III) We obtain 3 distinct bracelets: ab white, ac white, and ad white. The other three possibilities (bc, bd, cd) give the same bracelets as ad, ac, and ab (respectively).

We conclude that there are $4 + 4 + 3 = 11$ distinct bracelets in total.

B1. 2 We consider the expression from back to front. By taking $2017-2018-2019+2020$, we get zero. Then, we choose $2013 - 2014 - 2015 + 2016$, and so on, until $5 - 6 - 7 + 8$. For the first four numbers we choose $1 + 2 + 3 - 4$ to obtain a combined outcome of 2.

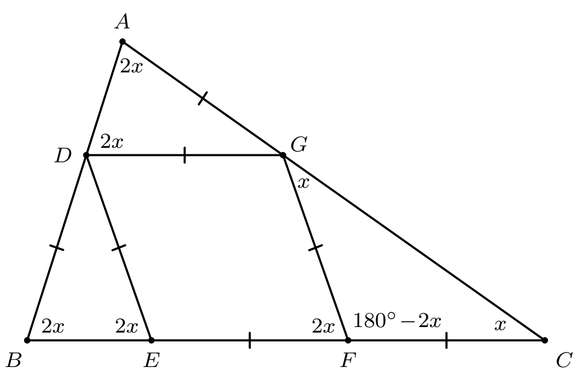
To conclude that this is optimal, it suffices to show that the outcome can never be 1, making 2 the smallest positive outcome possible. This is easy to see by observing that the outcome must always be *even*. Indeed, the outcome 2 is even and if we change any + into a − or conversely, the outcome will remain even if it was so before. All possible outcomes are therefore even.

B2. 36 First consider triangle CGF (see the figure). Let x be the angle at C (in degrees). Since this triangle is isosceles, the angle at G equals x as well. The angle at F is therefore $180° - 2x$. It follows that in the rhombus, the angle at F equals $180° - (180° - 2x) = 2x$.

Since DE and GF are parallel, it follows (corresponding angles) that in triangle BED the angle at E equals $2x$ as well. Since triangle BED is isosceles, also the angle at B equals $2x$.

In a similar way, we see that in triangle ADG the angle at D equals the angle at B, since BC and DG are parallel (corresponding angles). Finally, the angle at A equals the angle at D since triangle ADG is isosceles. So the angles at A and D both equal $2x$.

Since the angles of triangle ABC must sum to $180°$, we find $x + 2x + 2x = 180°$. This implies that $x = 36°$ and we conclude that the angle at C is 36 degrees.



B3. 5 Suppose that the numbers on Annemiek's note are a, b, and c, where $a < b < c$. Then the numbers on Bart's note, in increasing order, are $a + b$, $a + c$, and $b + c$. The last two of these are bigger than c and therefore bigger than any number on Annemiek's note. The number that is on both notes must therefore be $a + b$, which has to be equal to c.

So Annemiek has the numbers a, b, and $a + b$, and Bart has the numbers $a + b$, $2a + b$, and $a + 2b$. Looking at $3b$ and $3(a + b) = 3a + 3b$, we see that these are both bigger than any number on Bart's note. Annemiek's favourite number (the triple of which is on Bart's note), must therefore be a.

So the number $3a$ is one of the three numbers $a + b, 2a + b, a + 2b$ on Bart's note, and since a and b are different, it must be $a + b$. In other words: $b = 2a$. So Bart has the numbers $3a$, $4a$, and $5a$. Of these three numbers, only $5a$ can be equal to 25. Hence, $a = 5$ and this is Annemiek's favourite number.

B4. 10 Consider a number of adjacent coins, none of which is of denomination 3. At most one of these coins can be of denomination 2, because otherwise we have two coins of denomination 2 with only coins of denomination 1 in between them, which is impossible. Since at most one of the coins is of denomination 2, at most two of the coins can be of denomination 1, since otherwise we would have two adjacent coins of denomination 1. Therefore, the only possibilities for a stretch of adjacent coins of denominations 1 and 2 only are:

$$121,\ 12,\ 21,\ 2,\ 1.$$

A row of 2020 coins must therefore have multiple coins of denomination 3. Consider two coins of denomination 3 with no other coins of denomination 3 in between them. Between these two coins there must be at least three other coins. As we have just seen, this means that there are exactly three coins in between them and the only possibility is 121.

A row of 2020 coins therefore looks like this:

$$\text{Start-}\underbrace{312131213\cdots\cdots\cdots 31213}_{\text{repetition of the same pattern}}\text{-End},$$

where Start and End contain coins of denomination 1 and 2 only. So for Start and End the only possibilities are: 121, 12, 21, 2, 1, or no coins.

The length of the repeating part is a multiple of 4 plus one. The length must be less than $505 \cdot 4 + 1 = 2021$ and more than $503 \cdot 4 + 1 = 2013$, since otherwise Start and End together would have more than 6 coins. The length of the middle part is therefore $504 \cdot 4 + 1 = 2017$ and Start and End together have $2020 - 2017 = 3$ coins. Thus, we have 10 possibilities for Start and End: no coins in Start while End is 121, or vice versa (2 possibilities in total); Start being either 1 or 2, and End being 12 or 21 ($2 \cdot 2 = 4$ possibilities); Start being 12 or 21, and End being 1 or 2 ($2 \cdot 2 = 4$ possibilities).