First round
Dutch Mathematical Olympiad

Friday 27 January 2012

Solutions

A1. D) 12 Consider the left-hand figure. The cell marked by the star must contain a number that divides both 27 and 6. Hence this number is either 1 or 3. The first option implies that the cell marked by the double star contains the number 27. But 36 is not divisible by 27, so this option is off.

×		**		7
	24			56
		36	8	
*		27	6	
6	18			42

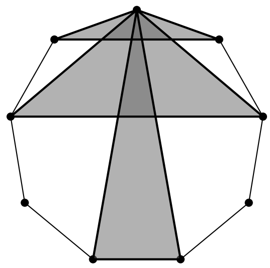
×	3	9	2	7
8	24	72	16	56
4	12	36	8	28
3	9	27	6	21
6	18	54	12	42

Using the second option, we can complete the full table and arrive at the right-hand figure. Clearly, 12 is the largest number occurring more than once.

A2. A) 11111 A 5-digit palindromic number is precisely determined by specifying the first three digits: the last digit then equals the first digit, and the second last digit equals the second digit. Therefore, the first twelve 5-digit palindromic numbers are:

$$\mathbf{100}01, \mathbf{101}01, \mathbf{102}01, \ldots, \mathbf{109}01, \mathbf{110}11, \mathbf{111}11.$$

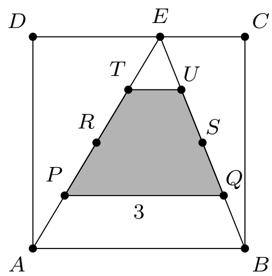
A3. B) 30 We start by counting the number of equilateral triangles: there are 3 of those. Next, we count the number of isosceles triangles that are not equilateral. Such a triangle has 9 possibilities for its apex. For each choice of the apex, there are 3 possibilities for its base. This gives a total of $9 \times 3 = 27$ possibilities. Hence, in total we have $3 + 27 = 30$ isosceles triangles.



A4. B) H, I and M Different letters may represent the same vertex of the dipiramid. The five vertices are $G = F = J$, $K = M$, H, I and L. Since H and I both have four neighbours, they represent vertices of the grey triangle. Also $K = M$ has four neighbours: I, H, L and $G = F = J$, so that must be the third vertex of the grey triangle.

A5. D) 32 Denote the number of blue socks in the drawer by b. To be sure of getting two *blue* socks, Frank must take at least 12 socks. Indeed, in the worst case he will start by picking the ten red socks. To be sure of getting at least two *red* socks, he must take at least $b + 2$ socks, because in the worst case he will first pick all b blue socks.
Since we know that this second number is twice the first number, we find that $b+2 = 2 \times 12 = 24$. This implies that $b = 22$. The total number of socks is therefore $22 + 10 = 32$.

A6. B) 4 Triangles EAB, EPQ, and ETU are similar since $|AE| : |PE| : |TE| = 4 : 3 : 1 = |BE| : |QE| : |UE|$ and $\angle AEB = \angle PEQ = \angle TEU$ (sas). This implies that $|AB| : |PQ| : |TU| = 4 : 3 : 1$. It follows that $|AB| = 4$ and hence the area of triangle ABE equals $\frac{1}{2} \times 4 \times 4 = 8$.
Triangles PQE and TUE must therefore have area $(\frac{3}{4})^2 \times 8 = \frac{9}{2}$ and $(\frac{1}{4})^2 \times 8 = \frac{1}{2}$, respectively, by similarity. It follows that the area of the quadrilateral $PQUT$ is equal to $\frac{9}{2} - \frac{1}{2} = 4$.

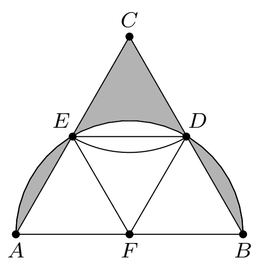


A7. D) 4 Because only two different outcomes occur, there can be no more than two different numbers on the cards. Indeed, if there would be three cards with different numbers, they would give three different outcomes when combined with a pair of the remaining three cards.
Let's denote the two numbers on the cards by a and b. Without loss of generality, we can assume that a occurs on at least three cards. Because $a+a+a$, $a+a+b$, and $a+b+b$ are different, the number b can occur only once.
There are two cases: the case $a+a+a=16, a+a+b=18$ and the case $a+a+a=18, a+a+b=16$. The first case is off because 16 is not a multiple of 3 (a is an integer). Hence $a=\frac{18}{3}=6$ and $b=16-12=4$. We conclude that 4 is the smallest number occurring on the cards.

A8. E) 1342 For any four consecutive terms a, b, c, d in the sequence, we have $d=a-1$. This follows from the fact that $b+c+d=(a+b+c)-1$ by the given property of the sequence. In other words: skipping forward in the sequence by three positions, decreases the current number by one. Hence in positions $3, 6, 9, \ldots, 2013$ $(=3+3\times 670)$ we find the numbers $2012, 2011, \ldots,$ $(2012-670=)$ 1342.

B1. $\frac{1}{2}$ The 5-digit numbers with product of the digits equal to 25, consist of two fives and three ones. There are a of these numbers, where a is the number of ways to place the three ones (because the positions of the fives is then fixed as well).
The 5-digit numbers with product of the digits equal to 15, consist of one three, one five, and three ones. There are b of those numbers, where b equals the number of ways to first place the three ones, and then place the digits 3 and 5 in one of two possible ways.
We see that $b=2a$, which implies that $\frac{a}{b}=\frac{1}{2}$.

B2. 6π Point F is the midpoint of AB (and the center of the semi-circle). Points D and E are the midpoints of BC and AC. They lie on the semicircle because triangles BDF and AEF are equilateral. Draw the arc between D and E of the circle with center C and radius $|CE|=6$. Because $\angle AFE=\angle EFD=60°$, the circular segments on top of AE and on top of DE are congruent. The circular segment below DE is congruent to the other two, because the circles around C and F have the same radius. It follows that the three grey regions together have the same area as the circle sector CED, which has an area of $\frac{1}{6}(\pi\cdot 6^2)=6\pi$.



B3. 48 and 60 Let a and b denote the number of cells in the length and width of the rectangle. We may assume that $a \geqslant b$. Therefore, the total number of cells in the rectangle is ab and the number of cells at the edges equals $2a+2b-4$. Given the fact that half of the cells are at the edge of the rectangle, we know that $ab=2(2a+2b-4)$. Rearranging terms gives $ab-4a-4b+16=8$. Factoring the left-hand side, we get $(a-4)(b-4)=8$.
Since a and b are positive integers and $a \geqslant b$, the only possibilities are $a-4=8, b-4=1$ and $a-4=4, b-4=2$ (note that $a-4$ and $b-4$ cannot be negative because in that case $b-4$ is at most -4, contradicting the fact that b is positive). We find $a=12, b=5$ or $a=8, b=6$. This gives two possible rectangles with $12\times 5=60$ and $8\times 6=48$ cells respectively.

B4. $\frac{11}{2}$ Using **Rule 1** we find: $32 \triangle 8=\frac{1}{2}+16 \triangle 8=\frac{2}{2}+8 \triangle 8=\frac{3}{2}+4 \triangle 8$.
From **Rule 2** with $y=2$ and $x=8$ it follows that : $4 \triangle 8=64 \triangle 2$.
Repeated application of **Rule 1** gives: $64 \triangle 2=\frac{1}{2}+32 \triangle 2=\frac{2}{2}+16 \triangle 2=\ldots=\frac{5}{2}+2 \triangle 2$.
Collecting these results and using **Rule 3**, we obtain:

$$32 \triangle 8=\frac{3}{2}+4 \triangle 8=\frac{3}{2}+64 \triangle 2=\frac{8}{2}+2 \triangle 2=\frac{11}{2}.$$