56th Dutch Mathematical Olympiad 2017

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2018 started with the first round in January 2017, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In this first round 10529 students from 340 secondary schools participated.

The 987 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 124 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 160 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 56th edition 2017.

The 30 most outstanding candidates of the Dutch Mathematical Olympiad 2017 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

In February a team of four girls was chosen from the training group to represent the Netherlands at the EGMO in Florence, Italy, from 9 until 15 April. The team brought home a silver medal, two bronze medals, and a honourable mention; a very nice achievement. For more information about the EGMO (including the 2018 paper), see www.egmo.org.

In March a selection test of three and a half hours was held to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Mersch, Luxemburg, from 27 until 29 April. The Dutch team received a gold medal, four silver medals and three bronze medals, and managed to get the highest total score of the Benelux countries, beaten only by guest country France. For more information about the BxMO (including the 2018 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2018 was selected by three team selection tests on 7, 8 and 9 June 2018, each lasting four hours. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Cluj-Napoca, from 30 June until 6 July.

For younger students the Junior Mathematical Olympiad was held in October 2017 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2018 in Romania consists of

- Nils van de Berg (18 years old)
  - bronze medal at BxMO 2017
  - gold medal at BxMO 2018
  - hon. mention at IMO 2017
- Szabi Buzogany (18 years old)
  - silver medal at BxMO 2018
- Thomas Chen (17 years old)
  - gold medal at BxMO 2017
  - silver medal at BxMO 2018
- Jovan Gerbscheid (15 years old)
  - silver medal at BxMO 2018
- Jippe Hoogeveen (15 years old)
- Matthijs van der Poel (17 years old)
  - bronze medal at BxMO 2016
  - bronze medal at BxMO 2017
  - observer C at IMO 2016
  - silver medal at IMO 2017

We bring as observer C the promising young student

- Richard Wols (15 years old)
  - bronze medal at BxMO 2018

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Jetze Zoethout (observer B), Utrecht University

First Round, January 2017

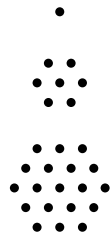
Problems

A-problems

1. In a certain year, August has only 4 Mondays and 4 Fridays. Which day of the week was 31 August that year?

A) Tuesday B) Wednesday C) Thursday
D) Saturday E) Sunday

2. We consider dotted hexagons with $1, 2, 3, \ldots$ dots on each side, see also the picture. The number of dots in such a hexagon is called a *hexagonal number*. The first hexagonal number is 1, the second is 7, and the third is 19.
Which of the following numbers is also a hexagonal number?



A) 81 B) 128 C) 144 D) 169 E) 187

3. Five suspects are arrested in a criminal investigation. Each of them makes one statement:

Eva: "We are all innocent."

Fatima: "Exactly one of us is innocent."

Kees: "Exactly one of us is guilty."

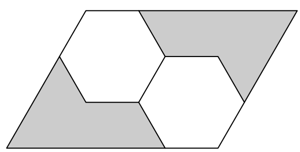
Manon: "At least two of us are innocent."

Mustafa: "At least two of us are guilty."

It turns out that those who are guilty lied, while those who are innocent told the truth. How many of the five suspects are guilty?

A) 1 B) 2 C) 3 D) 4 E) 5

4. Two regular hexagons share a side and are situated inside a parallelogram as indicated in the figure. The area of the parallelogram equals 1.
What is the area of the two grey areas combined?



A) $\frac{1}{3}$ B) $\frac{2}{5}$ C) $\frac{5}{12}$ D) $\frac{3}{7}$ E) $\frac{1}{2}$

5. In the expression below, the ten dots are replaced by ten distinct digits (0 to 9) in such a way that none of the resulting two-digit numbers starts with 0:

$$\frac{\cdot\cdot + \cdot\cdot + \cdot\cdot}{\cdot\cdot - \cdot\cdot}$$

What is the largest possible outcome we can obtain?

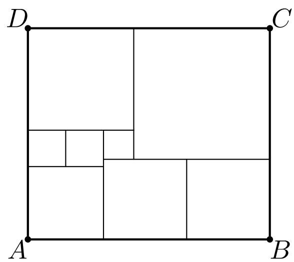
A) $\frac{255}{7}$ B) $\frac{219}{2}$ C) 116 D) 222 E) 255

6. A 100×100 table is filled with numbers. The bottom left cell contains the number 0. For every other cell V, we consider a route from the bottom left cell to V, where in each step we go one cell to the right or one cell up (not diagonally). If we take the number of steps and add the numbers from the cells along the route, we obtain the number in cell V. In the figure, you see a partially filled table. The number 15, for example, is obtained as $4 + (0 + 1 + 3 + 7) = 15$. What is the last digit of the number in the upper right cell of the 100×100 table?

$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	7	15	$\cdots$
1	3	7	$\cdots$
0	1	3	$\cdots$

A) 1 B) 3 C) 5 D) 7 E) 9

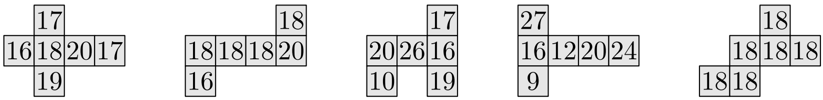
7. Rectangle $ABCD$ is divided into squares. The length of side AB is 16.



What is the length of side AD?

A) 13 B) $\frac{27}{2}$ C) 14 D) $\frac{29}{2}$ E) 15

8. Joep assigns the numbers 1 to 8 to the vertices of a cube (each vertex receiving a number different from the other vertices). For each face of the cube he adds the four numbers assigned to the vertices of that face and writes the resulting number on the face. Then, he cuts the cube open along some of the sides and flattens it out to obtain one of the five figures given below. Only one of these figures could represent Joep's cube.



Which figure could represent Joep's cube?

A) the first B) the second C) the third
D) the fourth E) the fifth

B-problems

The answer to each B-problem is a number.

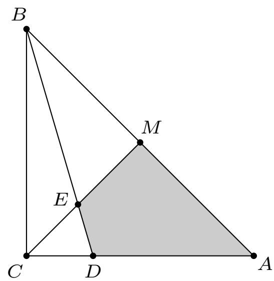
1. Isaac writes down a three digit number. None of its digits is a zero. Isaac gives his sheet with the number to Dilara, and below Isaac's number she writes down all three digit numbers that one can obtain by putting the digits of Isaac's number in a different order. Then she adds up all numbers on the sheet. The outcome is 1221.
What is the greatest number that Isaac could have written down?

2. There are two triples (a, b, c) of positive integers that satisfy the equations

$$\begin{aligned} ab + c &= 34, \\ a + bc &= 29. \end{aligned}$$

Which two triples are these?

3. Triangle ABC is an isosceles right angled triangle whose right angle is at C, with $|AC| = |BC| = 12$. Point M is the midpoint of side AB. A point D lies on side AC. Finally, point E is the intersection point of line segments CM and BD, see the figure.
If $|CD| = 3$, what is the area of quadrilateral $AMED$?



4. At a quiz you have to answer 10 questions. Each question is either difficult or easy. For a difficult question 5 points are being awarded for a correct answer and -1 point for an incorrect answer; for an easy question 3 points are being awarded for a correct answer and -1 point for an incorrect answer. Moreover, if you answer a question correctly, then the next question will be a difficult one; if you answer a question incorrectly, then the next question will be an easy one. You start with a difficult question.
How many distinct final scores are possible after 10 questions?

Solutions

A-problems

1. C) Thursday

2. D) 169

3. C) 3

4. E) $\frac{1}{2}$

5. D) 222

6. B) 3

7. D) $\frac{29}{2}$

8. E) the fifth

B-problems

1. 911

2. $(13, 2, 8)$ and $(5, 6, 4)$

3. $\frac{162}{5}$

4. 27

Second Round, March 2017

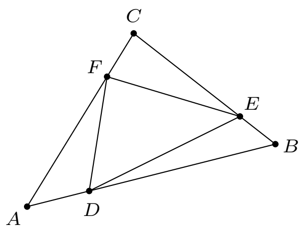
Problems

B-problems

The answer to each B-problem is a number.

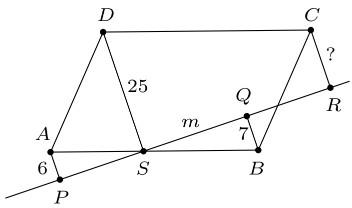
1. A finite sequence of consecutive positive integers is called *balanced* if it contains equally many multiples of three and multiples of five. An example of a sequence of length 7 that is not balanced is 30, 31, 32, 33, 34, 35, 36, because this sequence contains 3 multiples of three (namely 30, 33, and 36) and just 2 multiples of five (namely 30 and 35).
What is the maximal length of a balanced sequence of consecutive positive integers?

2. The area of a given triangle ABC equals 40. Point D on side AB satisfies $|BD| = 3 \cdot |AD|$. Point E on side BC satisfies $|CE| = 3 \cdot |BE|$. Point F on side CA satisfies $|AF| = 3 \cdot |CF|$. Determine the area of triangle DEF.



3. In math class, a student has written down a sequence of 16 numbers on the blackboard. Below each number, a second student writes down how many times that number occurs in the sequence. This results in a second sequence of 16 numbers. Below each number of the second sequence, a third student writes down how many times that number occurs in the second sequence. This results in a third sequence of numbers. In the same way, a fourth, fifth, sixth, and seventh student each construct a sequence from the previous one. Afterwards, it turns out that the first six sequences are all different. The seventh sequence, however, turns out to be equal to the sixth sequence.
Give one sequence that could have been the sequence written down by the first student.

4. A parallelogram $ABCD$ is intersected by a line m. From each of the four vertices A, B, C, and D we draw a perpendicular to m. The four feet are P, Q, R, and S, respectively. Point S is also the intersection of line m and AB. The lengths of line segments AP, BQ, and DS are 6, 7, and 25, respectively.
What is the length of CR?
Be careful: the figure is not drawn to scale.



5. Simon has 2017 blue blocks that are numbered from 1 up to and including 2017. He also has 2017 yellow blocks that are numbered from 1 up to and including 2017. Simon wants to arrange his 4034 blocks in a row, in such a way that, for every $k = 1, 2, \ldots, 2017$, the following conditions are met:

- to the left of blue block number k there are k or more yellow blocks;
- to the right of yellow block number k there are k or fewer blue blocks.

Determine all possible numbers for the 1000th block from the left in the row.

C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

1. You have 1000 tiles of each of the following five types:

1 0 1	1 1 0	1	1 0 0	0 1 0
A	B	C	D	E
0 1 0	1 1	1 0 1	0 1 0	0 1

You want to form a row of tiles such that the same sequence of zeroes and ones is formed on the top and the bottom. We will call this a *matching combination.* Consider, for example, the row 'DDC' consisting of three tiles of types D, D, and C, in that order. The top sequence is 1001001, while the bottom sequence is 010010101. Since the two sequences are not the same, the row of tiles is not a matching combination.

(a) Construct a matching combination using only tiles of type A, B, and C.

(b) Show that no matching combination using only tiles of types B, C, and D exists.

(c) Does a matching combination using only tiles of types B, C, D, and E exist?
If so, give an example. If not, prove that such a combination does not exist.

2. A *multi-square* is a number obtained by concatenating two or more square two-digit numbers. (A two-digit number is not allowed to start with digit 0). For example, since 16 and 25 are squares, 1625 is a multi-square.

(a) Determine all four-digit multi-squares whose first and last digit are equal.

(b) Determine all six-digit multi-squares that are themselves squares.

Solutions

B-problems

1. 11

2. $\frac{35}{2}$

3. Multiple solutions, e.g.: 0, 1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8

4. 12

5. 500

C-problems

C1. (a) A matching combination is 'BABC' having a top row and bottom row equal to 1101011101.
All other solutions are of the form 'BABCBABC... BABC'.

(b) Suppose that we have a matching combination using only tiles of types B, C, and D.

The top row of these tiles always starts with a 1. In a matching combination, the bottom row must therefore start with a 1 as well. This rules out the first tile being of type D. Type C is ruled out as well, since otherwise the second tile must have a top row starting with 0. The first tile of a matching combination must therefore be of type B.

The next tile must have a bottom row starting with 0. Hence, it must be of type D. After that, we again require a tile with a bottom row starting with a 0. That is, we again require a tile of type D. This continues indefinitely. After depleting our supply of 1000 tiles of type D, we still need another tile having a bottom row starting with a 0. It follows that we cannot complete our row to a matching combination: a contradiction.

We conclude that there is no matching combination using only tiles of types B, C, and D.

The proof of part (c) will give an alternative proof for part (b).

(c) First, consider the number of 1's in the top row and bottom row of each type of tile. Tiles of type B, D, and E have equal numbers of

1's in the top and bottom row. Tiles of type C have more 1's in the bottom row than in the top row. Since a matching combination must have equal numbers of 1's in both rows, it cannot contain tiles of type C.

A matching combination can therefore only contain tiles of types B, D, and E. Tiles of type D have an equal number of digits in the top row and bottom row (three digits). In contrast, tiles of type B and E have more digits in their top row than in their bottom row. Since any matching combination has equal numbers of digits in both rows, it can contain tiles of neither type B nor type E.

A matching combination can therefore only contain tiles of type D. But that is also not possible since tiles of type D have a top row starting with 1 and a bottom row starting with 0. We conclude that no matching combination exists containing only tiles of types B, C, D, and E.

C2. (a) No two-digit square ends in a 2, 3, 7, or 8. Also, no two-digit square starts with a 5 or 9. Hence, we only need to consider multi-squares with first and last digit equal to 1, 4, or 6.

In the case that the first and last digit are 1, the first square must be 16 and the second square must be 81. This yields the multi-square 1681. In the case that the first and last digit equal 4, the first square must be 49 and the second square must be 64. This yields the multi-square 4964. In the case that the first and last digit equal 6, the first square must be 64 and the second square must be 16 or 36. This yields two multi-squares: 6416 and 6436.

In total, there are four multi-squares having the same first and last digit: 1681, 4964, 6416, and 6436.

(b) Let K be a six-digit multi-square, say $K = \overline{abcdef}$, where a, b, c, d, e, and f are the six digits of K. The fact that K is a multi-square means that $\overline{ab}$, $\overline{cd}$, and $\overline{ef}$ are two-digit squares, hence equal to 16, 25, 36, 49, 64, or 81. We want K to be a square, say $K = n^2$. To determine all solutions, we consider the different cases for $\overline{ab}$.

$\boxed{\overline{ab} = 16}$ Since $K > 160000$, we must have $n > 400$. Write $n = 400 + x$, where x is a positive integer. We have $K = (400 + x)^2 = 160000+800x+x^2$. Since $K < 170000$, it follows that $800x < 10000$ and therefore $x \leq 12$.
Observe that the last two digits $\overline{ef}$ of $K = 160000 + 800x + x^2$ are equal to the last two digits of x^2. Since the last two digits of $10^2 = 100$, $11^2 = 121$, and $12^2 = 144$ do not form a square,

the candidates $x = 10, 11, 12$ are ruled out. Since $1^2 = 1$, $2^2 = 4$, and $3^2 = 9$ have only one digit, also the candidates $x = 1, 2, 3$ are ruled out.
We consider the remaining candidates $x = 4, 5, 6, 7, 8, 9$. In these cases, x^2 is a two-digit number, which implies that $\overline{cd} = 8 \cdot x$. This is a square only in the case $x = 8$. Hence, we find one solution: $408^2 = 166464$.

$\boxed{\overline{ab} = 25}$ Similarly to the previous case, we can write $n = 500 + x$, where x is a positive integer. We have $K = (500 + x)^2 = 250000 + 1000x + x^2$. Since $K < 260000$, it follows that $1000x < 10000$ and hence $x \leq 9$.
We observe that for every possible choice of x, the digit d of K will be equal to 0. Hence, there are no solutions.

$\boxed{\overline{ab} = 36}$ We write $n = 600 + x$, where x is a positive integer. We have $K = (600 + x)^2 = 360000 + 1200x + x^2$. Since $K < 370000$, it follows that $1200x < 10000$ and hence $x \leq 8$. Also, we have $x \geq 4$ as otherwise digit e will be equal to 0.
For the remaining candidates $x = 4, 5, 6, 7, 8$ we obtain $\overline{cd} = 12 \cdot x$. However, this is not a square for any of the possible choices for x. Hence, there are no solutions.

$\boxed{\overline{ab} = 49}$ We write $n = 700 + x$, where x is a positive integer. We have $K = (700 + x)^2 = 490000 + 1400x + x^2$. Since $K < 500000$, it follows that $1400x < 10000$ and hence $x \leq 7$. As in the previous case, we also have $x \geq 4$ because digit e cannot be 0.
For the remaining candidates $x = 4, 5, 6, 7$ we see that $\overline{cd} = 14 \cdot x$ is not a square. Hence, there are no solutions.

$\boxed{\overline{ab} = 64}$ We write $n = 800 + x$, where x is a positive integer. We have $K = (800 + x)^2 = 640000 + 1600x + x^2$. Since $K < 650000$, it follows that $1600x < 10000$ and hence $x \leq 6$. Again, we also have $x \geq 4$.
Of the remaining candidates $x = 4, 5, 6$, the number $\overline{cd} = 16 \cdot x$ is a square only for $x = 4$. Thus we find one solution $804^2 = 646416$.

$\boxed{\overline{ab} = 81}$ We write $n = 900 + x$, where x is a positive integer. We have $K = (900 + x)^2 = 810000 + 1800x + x^2$. Since $K < 820000$, it follows that $1800x < 10000$ and hence $x \leq 5$. Again we also have $x \geq 4$.
For the remaining candidates $x = 4, 5$, the number $\overline{cd} = 18 \cdot x$ is not a square. Hence, there are no solutions.

We conclude that, in total, there are two six-digit multi-squares that are themselves a square: 166464 and 646416.

Final Round, September 2017

Problems

1. We consider positive integers written down in the (usual) decimal system. Within such an integer, we number the positions of the digits from left to right, so the leftmost digit (which is never a 0) is at position 1.

An integer is called *even-steven* if each digit at an *even* position (if there is one) is greater than or equal to its neighbouring digits (if these exist).

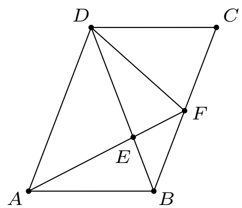
An integer is called *oddball* if each digit at an *odd* position is greater than or equal to its neighbouring digits (if these exist).

For example, 3122 is oddball but not even-steven, 7 is both even-steven and oddball, and 123 is neither even-steven nor oddball.

(a) Prove: every oddball integer greater than 9 can be obtained by adding two oddball integers.

(b) Prove: there exists an oddball integer greater than 9 that cannot be obtained by adding two even-steven integers.

2. A parallelogram $ABCD$ with $|AD| = |BD|$ has been given. A point E lies on line segment BD in such a way that $|AE| = |DE|$. The (extended) line AE intersects line segment BC in F. Line DF is the angle bisector of angle CDE.
Determine the size of angle ABD.



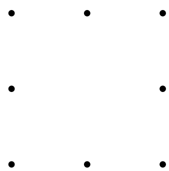
3. Six teams participate in a hockey tournament. Each team plays exactly once against each other team. A team is awarded 3 points for each game they win, 1 point for each draw, and 0 points for each game they lose. After the tournament, a ranking is made. There are no ties in the list. Moreover, it turns out that each team (except the very last team) has exactly 2 points more than the team ranking one place lower.
Prove that the team that finished fourth won exactly two games.

4. If we divide the number 13 by the three numbers 5, 7, and 9, then these divisions leave remainders: when dividing by 5 the remainder is 3, when dividing by 7 the remainder is 6, and when dividing by 9 the remainder is 4. If we add these remainders, we obtain $3 + 6 + 4 = 13$, the original number.

(a) Let n be a positive integer and let a and b be two positive integers smaller than n. Prove: if you divide n by a and b, then the sum of the two remainders never equals n.

(b) Determine all integers $n > 229$ having the property that if you divide n by 99, 132, and 229, the sum of the three remainders is n.

5. The eight points below are the vertices and the midpoints of the sides of a square. We would like to draw a number of circles through the points, in such a way that each pair of points lie on (at least) one of the circles. Determine the smallest number of circles needed to do this.



Solutions

1. An integer for which the digits (from left to right) are $c_1, c_2, \ldots, c_k$ will be denoted by $\overline{c_1c_2\ldots c_k}$.

(a) Let $n = \overline{c_1c_2\ldots c_k}$ be an oddball integer greater than 9 (hence $k \geq 2$). We will show that n is indeed the sum of two oddball integers.

If $c_2 \geq 1$, then we can write n as the sum of the following two oddball integers: $\overline{10\ldots0}$ ($k-2$ zeros) and $\overline{c_1(c_2-1)c_3\ldots c_k}$.

If $c_2 = 0$ and $c_1 \geq 2$, then we can write n as the sum of the following two oddball integers: $\overline{10\ldots0}$ ($k-1$ zeros) and $\overline{(c_1-1)c_2c_3\ldots c_k}$.

If $n = \overline{10\ldots0}$, then we can write n as the sum of the following two oddball integers: $\overline{1}$ and $\overline{9\ldots9}$ ($k-1$ nines).

The last case is the case in which $c_1 = 1$, $c_2 = 0$ and not all digits $c_3, \ldots, c_k$ are equal to 0. Let c_t be a digit unequal to 0, with $t \geq 3$ as small as possible. Hence, $n = \overline{10\ldots0c_t\ldots c_k}$ with $c_t \geq 1$.

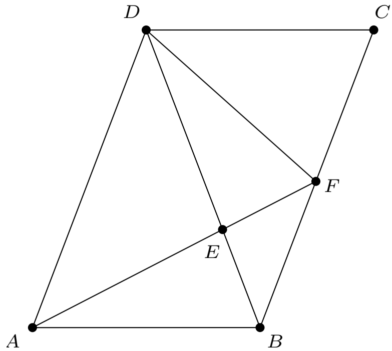
Because n is oddball and $c_{t-1} = 0 < 1 \leq c_t$, we find that t must be *odd*. We can now write n as the sum of the integers $\overline{10\ldots0}$ ($k-1$ zeros) and $m = \overline{c_tc_{t+1}\ldots c_k}$. Because t is odd, the digits at the odd positions of m are also at odd positions of n. Therefore, these digits are greater than or equal to their neighbouring digits (because n is oddball), which yields that m is oddball.

(b) The integer $n = 109$ is oddball, but it is not the sum of two even-steven integers. We will prove this by contradiction. Suppose that $n = p + q$ for some even-steven integers p and q. We will show that this leads to a contradiction.

First observe that the integers 100 to 108 are not even-steven. Hence, both p and q must be smaller than 100, and hence both are also greater than 9. In other words, p and q have exactly two digits. Suppose $p = \overline{ab}$ and $q = \overline{cd}$. The equation $p + q = 109$ now yields $b + d = 9$ (because $b + d < 19$) and $a + c = 10$. Hence, $b + d < a + c$, which implies that either $b < a$ or $d < c$ (or both). In the first case p is not even-steven and in the second case q is not even-steven. This contradicts the assumption that p and q are even-steven.

We conclude that the oddball integer 109 cannot be written as the sum of two even-steven integers.

2. Since AED is an isosceles triangle, angles $\angle EDA$ and $\angle DAE$ are equal. In turn, these angles are equal to angles $\angle EBF$ and $\angle BFE$ (alternate interior angles). This implies that triangle BFE is isosceles as well, with $|BE| = |EF|$.



Comparing triangles ABE and DFE, we see that $|AE| = |DE|$ and $|BE| = |FE|$. Since $\angle BEA$ and $\angle FED$ are a pair of opposite angles, they have the same size. It follows that triangles ABE and DFE are congruent (SAS).

From the fact that ABE and DFE are congruent it follows that $|DF| = |AB|$. Since $ABCD$ is a parallelogram, we also have $|AB| = |CD|$. It follows that triangle CDF is isosceles as well (with apex D).

On the one hand, this implies that $\angle FCD = \angle DFC$. On the other hand, we know that triangle DBC is isosceles (since $|BD| = |AD| = |BC|$), which implies that $\angle FCD = \angle CDB = 2 \cdot \angle CDF$ since DF is the angle bisector of $\angle CDB$.

Altogether, we have $\angle DFC = \angle FCD = 2 \cdot \angle CDF$. Since the angles in any triangle sum to 180 degrees, we also know that

$$180° = \angle DFC + \angle FCD + \angle CDF = 5 \cdot \angle CDF.$$

From this, it follows that $\angle CDF = \frac{1}{5} \cdot 180° = 36°$, and hence $\angle FCD = 2 \cdot \angle CDF = 72°$. Since triangle DBC is isosceles, also $\angle CDB = 72°$ holds. Using alternating interior angles, we now find that $\angle ABD = \angle CDB = 72°$.

3. Let the scores of the six teams be s, $s+2$, $s+4$, $s+6$, $s+8$, and $s+10$. Let T be the total number of awarded points, so that $T = 6s + 30$. It follows that the total number of points is a multiple of six.

The number of games played equals $\frac{6 \cdot 5}{2} = 15$. Let g be the number of games that ended in a draw. A game that ends in a draw results in $1 + 1 = 2$ awarded points and every other game results in $3 + 0 = 3$ awarded points. Therefore, the total number of awarded points equals $T = g \cdot 2 + (15 - g) \cdot 3 = 45 - g$.

From $T = 45 - g$ it follows that $30 \leq T \leq 45$ because the number of draws satisfies $0 \leq g \leq 15$. Since T is a multiple of six, this leaves the following possibilities: $T = 30$, $T = 36$, and $T = 42$.

If $T = 30$, we have $g = 45 - 30 = 15$. But then all games must have ended in a draw and all teams must have the same score. Hence, the case $T = 30$ is ruled out.

If $T = 36$, then $g = 45 - 36 = 9$ and $s = \frac{T-30}{6} = 1$. The six scores are therefore $1, 3, 5, 7, 9, 11$. The team that scored 1 point must have lost 4 games (and played one draw). The team that scored 3 points must have lost at least 2 games (at most 3 games were not lost). The team that scored 11 points must have won at least 3 games (otherwise the score is at most $3+3+1+1+1 = 9$), so apart from the teams with scores 1 and 3 at least one other team has lost a game. In total, at least $4 + 2 + 1 = 7$ games ended in a loss for some team, contradicting the fact that $15 - 9 = 6$ games did not end in a draw. This rules out the case $T = 36$.

Finally, we consider that case $T = 42$. The six scores are $2, 4, 6, 8, 10, 12$ and we have $g = 3$. Since the total number of points obtained from won games is a multiple of three, the six teams must have received at least $2, 1, 0, 2, 1, 0$ points from draws, respectively. In total, exactly $2 \cdot 3 = 6$ points are awarded in games that ended in a draw. Hence, since $2 + 1 + 0 + 2 + 1 + 0 = 6$, the six teams have received exactly the mentioned numbers of points from draws. In particular, the team ending in the fourth place (with 6 points), was involved in 0 draws and must have won exactly two games.

4. (a) Let r be the remainder upon dividing n by a. We will first prove that $r < \frac{n}{2}$. If $2a \leq n$, this follows from the fact that $r < a$. If $2a > n$, we have $r = n - a$ (since we already knew that $a < n$), which implies that $r = n - a < n - \frac{n}{2} = \frac{n}{2}$.

For the same reasons, the remainder upon dividing n by b is smaller than $\frac{n}{2}$.

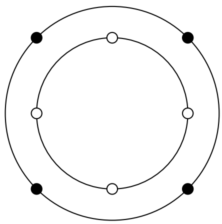
It follows that the two remainders obtained by dividing n by a and b add up to a number smaller than n.

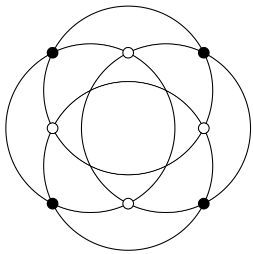
(b) Let r, s, and t be the remainders upon dividing n by 99, 132, and 229, respectively. The number $n - t$ is a multiple of 229 and nonzero since $n > 229 > t$. We know that $r + s + t = n$, and hence $n - t = r + s$. We can conclude that $r + s$ is a positive multiple of 229. Since $99 + 132 < 2 \cdot 229$, we have $r + s < 2 \cdot 229$, which implies that we must have $r + s = 229$.

Since $r \leq 98$ and $s \leq 131$, the fact that $r + s = 229$ implies that $r = 98$ and $s = 131$. Therefore, the number $n + 1$ is divisible by both 99 and 132, and hence by their least common multiple $\text{lcm}(99, 132) = \text{lcm}(9 \cdot 11, 3 \cdot 4 \cdot 11) = 4 \cdot 9 \cdot 11 = 396$. Also, from $n = 229 + t$ and $t < 229$ we deduce that $n+1 \leq 458$. It follows that the only possibility is $n + 1 = 396$, hence $n = 395$.

When $n = 395$ the three remainders are $r = 98$, $s = 131$, and $t = 166$, and indeed satisfy the equation $n = r + s + t$.

5. Four of the eight points are coloured black and the other four points are coloured white in the way indicated in the figure on the left. The circle through the four black points is denoted C_1 and the circle through the four white points is denoted C_2. If two points lie on a circle C, we say that C *covers* that pair of points.





Circle C_1 covers all pairs of black points and circle C_2 covers all pairs of white points. It is easy to check that each of the $4 \cdot 4 = 16$ pairs consisting of a white point and a black point is covered by one of the four circles in the figure on the right. It follows that the six circles form a solution.

We will now prove that five or fewer circles do not suffice. First observe that any circle passing through more than two black points must be equal to C_1 and that any circle passing through more than two white points must be equal to C_2. Indeed, a circle is already determined by three points.

A circle passing through 2 or fewer black points covers at most one of the $\frac{4 \cdot 3}{2} = 6$ pairs of black points. A solution consisting of only five circles must therefore contain circle C_1 (since otherwise at most 5 pairs of black points are covered). In the same way we see that such a solution must contain circle C_2.

Each of the remaining three circles in the (hypothetical) solution contains at most 2 black points and at most 2 white points. Such a circle covers at most $2 \cdot 2 = 4$ pairs consisting of a white and a black point. In total, the five circles can therefore cover at most $0 + 0 + 3 \cdot 4 = 12$ such pairs, while there are 16 to be covered. The five circles can therefore not form a correct solution after all. We conclude that the smallest number of circles in a solution is 6.

BxMO Team Selection Test, March 2018

Problems

1. We have 1000 balls in 40 different colours, 25 balls of each colour. Determine the smallest n for which the following holds: if you place the 1000 balls in a circle, in any arbitrary way, then there are always n adjacent balls which have at least 20 different colours.

2. Let $\triangle ABC$ be a triangle of which the side lengths are positive integers which are pairwise coprime. The tangent in A to the circumcircle intersects line BC in D. Prove that $|BD|$ is not an integer.

3. Let p be a prime number. Prove that it is possible to choose a permutation $a_1, a_2, \ldots, a_p$ of $1, 2, \ldots, p$ such that the numbers a_1, a_1a_2, $a_1a_2a_3$, $\ldots$, $a_1a_2a_3\cdots a_p$ all have different remainder upon division by p.

4. In a non-isosceles triangle $\triangle ABC$ we have $\angle BAC = 60°$. Let D be the intersection of the angular bisector of $\angle BAC$ with side BC, O the centre of the circumcircle of $\triangle ABC$ and E the intersection of AO and BC. Prove that $\angle AED + \angle ADO = 90°$.

5. Let n be a positive integer. Determine all positive real numbers x satisfying

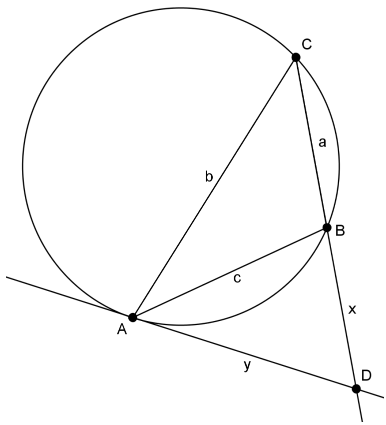
$$nx^2 + \frac{2^2}{x+1} + \frac{3^2}{x+2} + \ldots + \frac{(n+1)^2}{x+n} = nx + \frac{n(n+3)}{2}.$$

Solutions

1. Consider the circle in which the 25 balls of one colour are all next to each other. To get at least 20 different colours, you have to take at least 18 of these groups plus one ball one one side and one ball on the other side of these groups. In total, you need at least $18 \cdot 25 + 2 = 452$ adjacent balls. Hence, $n \geq 452$.

Now we will prove that 452 balls is enough. Consider an arbitrary circle of balls and all possible sets of consecutive balls having exactly 20 colours. (There exists at least one such a set: take an arbitrary ball and add balls on the left one by one, until you have exactly 20 colours.) Take such a set having a minimum number of balls. Suppose that the first ball is white. If there is another white ball in the set, then we could have removed the first ball to get a smaller set with the same number of colours, but less balls in total. This contradicts the minimality of our set. Therefore, no other ball is white. In particular, also the last ball of the set is not white; suppose it is black. In the same way, we find that no other ball in the set is black. So there are only one white ball, one black ball, and balls in 18 other colours, at most 25 of each colour. In total, there are at most $18 \cdot 25 + 2 = 452$ balls. Indeed, it is always possible to find a set of 452 consecutive balls in at least 20 different colours. We conclude that the minimum n is 452. $\square$

2. There are two different configurations. Without loss of generality, assume that B lies between D and C. Let $a = |BC|$, $b = |CA|$, $c = |AB|$, $x = |BD|$, and $y = |AD|$. Due to the alternate segment theorem we have $\angle BAD = \angle ACB = \angle ACD$, hence $\triangle ABD \sim \triangle CAD$ (AA), therefore $\frac{|BD|}{|AD|} = \frac{|AB|}{|CA|} = \frac{|AD|}{|CD|}$, or $\frac{x}{y} = \frac{c}{b} = \frac{y}{a+x}$. This yields $yc = bx$ and $ac + xc = by$, and hence $byc = b^2x$ and $ac^2 + xc^2 = byc$. Combining this, we obtain $b^2x = ac^2 + xc^2$, or $x(b^2 - c^2) = ac^2$.



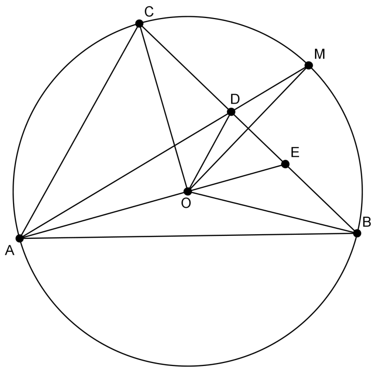
Suppose on the contrary that x is an integer. Then $b^2 - c^2$ is a divisor of ac^2. But we know $\gcd(b, c) = 1$, and hence $\gcd(b^2 - c^2, c) = \gcd(b^2, c) = 1$. Therefore, $b^2 - c^2$ is a divisor of a. This yields $b^2 - c^2 \leq a$. Note that $b^2 - c^2 > 0$, because $x(b^2 - c^2) = ac^2$; hence, $b - c > 0$ also holds. Therefore, $b^2 - c^2 = (b - c)(b + c) \geq 1 \cdot (b + c)$, because b and c are positive integers. Hence, $a \geq b + c$, which is contradicting the triangle inequality. Therefore, $x = |BD|$ cannot be an integer. $\square$

3. Let $b_i = a_1 a_2 \cdots a_i$, for $1 \le i \le p$. We will prove that it is possible to choose a permutation such that $b_i \equiv i \mod p$ for all i. For $i \ge 2$, the congruence $a_i \equiv b_i \cdot b_{i-1}^{-1} \mod p$ holds, if $b_{i-1} \not\equiv 0 \mod p$. Therefore, we now choose $a_1 = 1$ and $a_i \equiv i \cdot (i-1)^{-1} \mod p$ for $2 \le i \le p$. Now it is sufficient to prove that $a_i \not\equiv 1 \mod p$ for all $2 \le i \le p$ and $a_i \not\equiv a_j \mod p$ for all $2 \le j < i \le p$.

Suppose the contrary. Then $a_i \equiv 1 \mod p$ for certain $2 \le i \le p$. Then $i \cdot (i-1)^{-1} \equiv 1 \mod p$ holds, hence $i \equiv i - 1 \mod p$, or $0 \equiv -1 \mod p$. Because $p \ge 2$, this is a contradiction. Now suppose that $a_i \equiv a_j \mod p$ for certain $2 \le j < i \le p$. Then we have $i \cdot (i-1)^{-1} \equiv j \cdot (j-1)^{-1} \mod p$, hence $i(j-1) \equiv j(i-1) \mod p$, or $ij - i \equiv ij - j \mod p$, or $-i \equiv -j \mod p$. However, as $2 \le j < i \le p$, this is impossible.

We conclude that if we choose the a_i as above, all a_i are distinct, so that they indeed form a permutation of 1, 2, ..., p. Moreover, by definition we have that $a_1 a_2 \cdots a_i \equiv i \mod p$, therefore also the second condition is satisfied. $\square$

4. Let M be the other intersection point of AD with the circumcircle of $\triangle ABC$. Then M is the midpoint of the circle arc BC on which A does not lie. Now we have $\angle COM = \frac{1}{2}\angle COB = \angle CAB = 60°$. Moreover, we have $|OC| = |OM|$, hence $\triangle OCM$ is isosceles with apex angle equal to $60°$. This means that it is equilateral, hence $|CM| = |CO|$. Because OM is perpendicular to BC, this yields that M is the reflection of O in BC, and hence $\angle DOM = \angle DMO$.



Moreover, we have $\angle DMO = \angle AMO = \angle MAO$, because $|OA| = |OM|$. We now obtain $\angle ODE = 90° - \angle DOM = 90° - \angle DMO = 90° - \angle MAO = 90° - \angle DAE$. Therefore, $\angle ODE + \angle DAE = 90°$. In triangle ADE we have $180° = \angle DAE + \angle AED + \angle ODE + \angle ADO$, hence we conclude that $\angle AED + \angle ADO = 90°$. $\square$

5. For $1 \le i \le n$ we have

$$\tfrac{(i+1)^2}{x+i} = i + 1 + \tfrac{(i+1)^2 - (i+1)(x+i)}{x+i} = i + 1 + \tfrac{(i+1)(1-x)}{x+i},$$

so we can rewrite the left hand side of the equation to

$$nx^2 + 2 + 3 + \ldots + (n+1) + \tfrac{2(1-x)}{x+1} + \tfrac{3(1-x)}{x+2} + \ldots + \tfrac{(n+1)(1-x)}{x+n}.$$

We have $2+3+\ldots+(n+1) = \frac{1}{2}n(n+3)$, hence this sum cancels against $\frac{n(n+3)}{2}$ on the right hand side of the original equation. Moreover, we can move nx^2 to the other side, and write a separate factor $1-x$ in all fractions. The equation then becomes

$$(1-x)\cdot\left(\tfrac{2}{x+1}+\tfrac{3}{x+2}+\ldots+\tfrac{n+1}{x+n}\right) = nx - nx^2.$$

The right hand side, we can factor as $nx(1-x)$. Now, we observe that $x=1$ is a solution to this equation. If there were another solutions $x \neq 1$, then it would satisfy

$$\tfrac{2}{x+1}+\tfrac{3}{x+2}+\ldots+\tfrac{n+1}{x+n} = nx.$$

However, for $0<x<1$ we have

$$\tfrac{2}{x+1}+\tfrac{3}{x+2}+\ldots+\tfrac{n+1}{x+n} > \tfrac{2}{2}+\tfrac{3}{3}+\ldots+\tfrac{n+1}{n+1} = n > nx,$$

while for $x>1$ we have

$$\tfrac{2}{x+1}+\tfrac{3}{x+2}+\ldots+\tfrac{n+1}{x+n} < \tfrac{2}{2}+\tfrac{3}{3}+\ldots+\tfrac{n+1}{n+1} = n < nx.$$

Hence, there are no solutions with $x \neq 1$. We conclude that $x=1$ is the only solution, for all n. $\square$

IMO Team Selection Test 1, June 2018

Problems

1. Suppose a grid with $2m$ rows and $2n$ columns is given, where m and n are positive integers. You may place one pawn on any square of this grid, except the bottom left one or the top right one. After placing the pawn, a snail wants to undertake a journey on the grid. Starting from the bottom left square, it wants to visit every square exactly once, except the one with the pawn on it, which the snail wants to avoid. Moreover, it wants to finish in the top right square. It can only move horizontally or vertically on the grid.

On which squares can you put the pawn for the snail to be able to finish its journey?

2. Suppose a triangle $\triangle ABC$ with $\angle C = 90°$ is given. Let D be the midpoint of AC, and let E be the foot of the altitude through C on BD. Show that the tangent in C of the circumcircle of $\triangle AEC$ is perpendicular to AB.

3. Let $n \geq 0$ be an integer. A sequence $a_0, a_1, a_2, \dots$ of integers is defined as follows: we have $a_0 = n$ and for $k \geq 1$, a_k is the smallest integer greater than a_{k-1} for which $a_k + a_{k-1}$ is the square of an integer. Prove that there are exactly $\lfloor \sqrt{2n} \rfloor$ positive integers that cannot be written in the form $a_k - a_\ell$ with $k > \ell \geq 0$.

4. Let A be a set of functions $f \colon \mathbb{R} \to \mathbb{R}$. For all $f_1, f_2 \in A$ there exists a $f_3 \in A$ such that

$$f_1\big(f_2(y) - x\big) + 2x = f_3(x + y)$$

for all $x, y \in \mathbb{R}$. Prove that for all $f \in A$, we have

$$f\big(x - f(x)\big) = 0$$

for all $x \in \mathbb{R}$.

Solutions

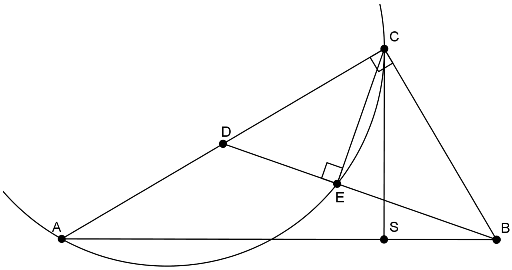
1. Number the rows from bottom to top by $1, 2, \ldots, 2m$, and the columns from left to right by $1, 2, \ldots, 2n$, so that the snail starts in square $(1, 1)$ and finishes in $(2m, 2n)$. Colour the squares of the grid like a chessboard, where (i, j) is coloured black if $i + j$ is even and white if $i + j$ is odd. Since $2m$ and $2n$ are even, the number of black squares equals that of white squares.

The snail starts and finishes on a black square, and alternates between black and white squares on its journey. Therefore, the number of black squares it visits is one larger than that of white squares, so the pawn must be on a white square (i.e. a square (i, j) with $i + j$ odd) for the snail to be able to finish its journey.

Let us show that if the pawn is on such a square, then the snail is always able to undertake its journey. Let $1 \leq k \leq m$ and $1 \leq l \leq n$ be such the pawn is either on row $2k - 1$ or row $2k$, and either on column $2l - 1$ or column $2l$. Then have the snail move all the way to the right on odd rows with a number less than $2k - 1$ (and then move one square up, which puts the snail on an even row), and all the way to the left on even rows with a number less than $2k - 1$ (and then move one square up, which puts the snail on an odd row again). This puts the snail on square $(2k - 1, 1)$. From here, while the snail is in a column with number less than $2l - 1$, have the snail move through each 2×2-block in the order: bottom left, top left, top right, bottom right. After moving one square to the right, this puts the snail on the bottom left square of the next 2×2-block. This puts the snail on square $(2k - 1, 2l - 1)$, which cannot be occupied by the pawn as $2k - 1 + 2l - 1$ is even. In this 2×2-block, the pawn is either on the top left or the bottom right square; in the former case, move the snail in the order: bottom left, bottom right, top right; in the latter case, move the snail in the order: bottom left, top left, top right. So in both cases, the snail ends up in the top right square of the 2×2-block. The remainder of the 2×2-blocks in rows $2k - 1$ and $2k$ can now be covered by moving in each of them in the order: top left, bottom left, bottom right, top right. This puts the snail on square $(2k, 2n)$, with all required squares to the bottom and left visited exactly once. The snail can now finish its journey by repeatedly doing the following: move one square up, move all the way to the left, move one square up, move all the way to the right. This puts the snail on square $(2m, 2n)$, as required.

Therefore, the snail is able to finish its journey if and only if the pawn is put on a square (i, j) with $i + j$ odd. $\square$

2. Let S be the intersection of the tangent with AB. Then we need to show that $\angle BSC = 90°$.



Since $\angle BCD = 90° = \angle CED$ we have $\triangle BCD \sim \triangle CED$, so $\frac{|DC|}{|DB|} = \frac{|DE|}{|DC|}$. As $|DA| = |DC|$ it follows that $\frac{|DA|}{|DB|} = \frac{|DE|}{|DA|}$. Together with $\angle BDA = \angle EDA$ this gives $\triangle ADE \sim \triangle BDA$ (SAS). Therefore $\angle ABD = \angle EAD$. Since SC is a tangent to the circumcircle of $\triangle AEC$ we also have that (by the tangent chord angle theorem) $\angle EAD = \angle EAC = \angle SCE$. So $\angle SCE = \angle ABD = \angle SBE$. Hence $SBCE$ is a cyclic quadrilateral. Therefore $\angle BSC = \angle BEC = 90°$. $\square$

3. Let $m = \lfloor\sqrt{2n}\rfloor$. We first show that the sequence of squares $a_0 + a_1, a_1 + a_2, \ldots$ is the sequence $(m+1)^2, (m+2)^2, \ldots$. After that, we will show that the differences $a_i - a_{i-1}$ form a sequence of consecutive even numbers and a sequence of consecutive odd numbers. We will use this to prove the given statement.

Note that $a_0 + a_1$ is the smallest square greater than $2n$. Therefore $a_0 + a_1 = (m+1)^2$. We show by induction on i that $a_{i-1} + a_i = (m+i)^2$. For $i = 1$, this is what we have just shown. Suppose that $a_{j-1}+a_j = (m+j)^2$. Then $a_{j-1} \geq \frac{(m+j-1)^2}{2}$ (since otherwise a_j could have been chosen in such a way that $a_{j-1} + a_j = (m+j-1)^2$), so

$$\begin{aligned} a_j &= (m+j)^2 - a_{j-1} \leq (m+j)^2 - \tfrac{(m+j-1)^2}{2} \\ &= \tfrac{2m^2+4mj+2j^2-(m^2+2mj+j^2+1-2m-2j)}{2} \\ &= \tfrac{m^2+2mj+j^2-1+2m+2j}{2} < \tfrac{m^2+2mj+j^2+1+2m+2j}{2} = \tfrac{(m+j+1)^2}{2}. \end{aligned}$$

We deduce that $a_j + a_{j+1} \leq (m+j+1)^2$. Moreover, we have $a_j + a_{j+1} > a_{j-1} + a_j = (m+j)^2$, so $a_j + a_{j+1} = (m+j+1)^2$, which completes the induction.

Define $b_i = a_i - a_{i-1}$ for all $i \geq 1$. Then we have

$$\begin{aligned} b_{i+2} - b_i &= a_{i+2} - a_{i+1} - a_i + a_{i-1} \\ &= (a_{i+2} + a_{i+1}) + (a_i + a_{i-1}) - 2(a_{i+1} + a_i) \\ &= (m+i+2)^2 + (m+i)^2 - 2(m+i+1)^2 \\ &= (m+i)^2 + 4(m+i) + 4 + (m+i)^2 - 2(m+i)^2 - 4(m+i) - 2 \\ &= 2 \end{aligned}$$

for all $i \geq 1$. Hence

$$(b_1, b_3, b_5, \ldots) = (b_1, b_1 + 2, b_1 + 4, \ldots),$$
$$(b_2, b_4, b_6, \ldots) = (b_2, b_2 + 2, b_2 + 4, \ldots).$$

We have $b_1 + b_2 = (a_2 - a_1) + (a_1 - a_0) = (a_2 + a_1) - (a_1 + a_0) = (m+2)^2 - (m+1)^2 = 2m+3$. In particular, b_1 and b_2 have distinct parities. So we can write every integer that is at least b_1 and has the same parity as b_1 as $a_k - a_{k-1}$ for some k. The same is true for every integer that is at least b_2 and has the same parity as b_2. All integers of the form $a_k - a_\ell$ with $k \geq \ell + 2$ are at least $b_k + b_{k-1} \geq b_1 + b_2$ and therefore both greater than b_1 and greater than b_2. Hence this does not give us any new numbers of the form $a_k - a_\ell$ with $k > \ell \geq 0$.

We deduce that the numbers not of the form $a_k - a_\ell$ with $k > \ell \geq 0$ are precisely those that either are less than b_1 and have the same parity as b_1, or are less than b_2 and have the same parity as b_2. There are $\left\lfloor \frac{b_1-1}{2} \right\rfloor + \left\lfloor \frac{b_2-1}{2} \right\rfloor$ of those. Note that the argument of exactly one of the floors is an integer. Hence we can rewrite the above as $\frac{b_1-1}{2} + \frac{b_2-1}{2} - \frac{1}{2} = \frac{b_1+b_2-3}{2} = \frac{2m}{2} = m$.

We conclude that there are exactly $m = \lfloor \sqrt{2n} \rfloor$ positive integer which cannot be written in the required form. $\square$

4. Substituting $x = 0$ gives $f_1\big(f_2(y)\big) = f_3(y)$ therefore the f_3 corresponding to $f_1, f_2 \in A$ is the composition $f_3(x) = f_1(f_2(x))$. Substituting $x = -y$ now gives that for all $f_1, f_2 \in A$ we have

$$f_1\big(f_2(y) + y\big) - 2y = f_3(0) = f_1(f_2(0))$$

for all $y \in \mathbb{R}$.

Substituting $x = f_2(y)$ now gives that for all $f_1, f_2 \in A$, there exists $f_3 \in A$ with

$$f_1(0) + 2f_2(y) = f_3\big(f_2(y) + y\big)$$

for all $y \in \mathbb{R}$. We have already seen that we can write $f_3(f_2(y) + y)$ as $2y + f_3(f_2(0))$. Hence

$$2f_2(y) = 2y + f_3(f_2(0)) - f_1(0)$$

for all $y \in \mathbb{R}$. We deduce that there exists a $d \in \mathbb{R}$, independent of y, such that $f_2(y) = y + d$ for all $y \in \mathbb{R}$.

Hence $f \in A$ is of the form $f(x) = x + d$ with d a constant. Therefore we have $f\big(x - f(x)\big) = f\big(x - (x + d)\big) = f(-d) = -d + d = 0$. $\square$

IMO Team Selection Test 2, June 2018

Problems

1. (a) If $c(a^3+b^3) = a(b^3+c^3) = b(c^3+a^3)$ with a, b, c positive real numbers, does $a = b = c$ necessarily hold?

(b) If $a(a^3+b^3) = b(b^3+c^3) = c(c^3+a^3)$ with a, b, c positive real numbers, does $a = b = c$ necessarily hold?

2. Find all positive integers n for which there exists a positive integer k such that for every positive divisor d of n, the number $d - k$ is also a (not necessarily positive) divisor of n.

3. Let ABC be an acute triangle, and let D be the foot of the altitude through A. On AD, there are distinct points E and F such that $|AE| = |BE|$ and $|AF| = |CF|$. A point $T \neq D$ satisfies $\angle BTE = \angle CTF = 90°$. Show that $|TA|^2 = |TB| \cdot |TC|$.

4. In the classroom of at least four students the following holds: no matter which four of them take seats around a round table, there is always someone who either knows both of his neighbours, or does not know either of his neighbours. Prove that it is possible to divide the students into two groups such that in one of them, all students know one another, and in the other, none of the students know each other.

(Note: if student A knows student B, then student B knows student A as well.)

Solutions

1. (a) We claim that $(a, b, c) = (2, 2, -1 + \sqrt{5})$ satisfies the given equalities. As all real numbers in this triple are positive, and $2 \neq -1 + \sqrt{5}$, the answer to the first question is "no".

Note that $c^3 = (-1 + \sqrt{5})^3 = -1 + 3 \cdot \sqrt{5} - 3 \cdot 5 + 5\sqrt{5} = -16 + 8\sqrt{5}$. Then we have $c(a^3 + b^3) = (-1 + \sqrt{5}) \cdot 2 \cdot 8 = -16 + 16\sqrt{5}$ and $a(b^3 + c^3) = b(c^3 + a^3) = 2 \cdot (-16 + 8\sqrt{5} + 8) = -16 + 16\sqrt{5}$, so the given equalities are indeed satisfied.

(b) We show that if the given equalities are satisfied, that then $a = b = c$. Without loss of generality, we assume that $a \geq b, c$. Then we have

$$a(a^3 + b^3) \geq b(a^3 + b^3) \geq b(c^3 + b^3) = a(a^3 + b^3),$$

so equality holds everywhere. Now $a = b$ follows from $a^3 + b^3$ being positive, and $a = c$ follows from b being positive. Therefore $a = b = c$.

$\square$

2. If n is either 1 or a prime number, then the positive divisors of n are 1 and n (which coincide if $n = 1$). In this case we can take $k = n + 1$ and note that $1 - (n + 1) = -n$ and $n - (n + 1) = -1$ are also divisors of n. Hence if $n = 1$ or if n is a prime number, the given property is satisfied. If $n = 4$, the positive divisors are 1, 2, and 4. Taking $k = 3$, we see that -2, -1, 1 are divisors of 4, so $n = 4$ also has the given property. If $n = 6$, the positive divisors are 1, 2, 3, and 6. Taking $k = 4$, we see that -3, -2, -1, and 2 are divisors of 6, so $n = 6$ has the given property. Hence if $n \leq 6$ or if n is a prime number, n has the given property.

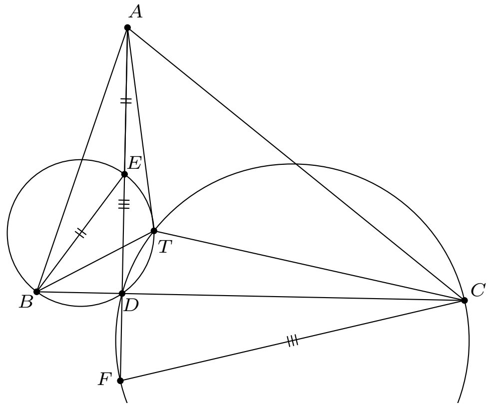
Now suppose that $n > 6$ is composite. Suppose that k is a positive integer such that for every positive divisor d of n, the number $d - k$ is also a divisor of n. Since n is a positive divisor of n, it follows that $n - k$ is a divisor of n. As the next largest divisor of n is at most $\frac{1}{2}n$, it follows that $n - k \leq \frac{1}{2}n$, and therefore that $k \geq \frac{1}{2}n$. Moreover, 1 is a positive divisor of n, so $1 - k$ is a divisor of n. Note that $1 - k \leq 1 - \frac{1}{2}n$. As $n > 6$, we have $\frac{1}{6}n > 1$, so $\frac{1}{2}n - \frac{1}{3}n > 1$, and therefore $-\frac{1}{3}n > 1 - \frac{1}{2}n$. The only divisors that are at most $1 - \frac{1}{2}n$ are therefore $-n$ and (if n is even) $-\frac{1}{2}n$. Therefore $1 - k = -n$ or $1 - k = -\frac{1}{2}n$.

In the latter case, we have $k = \frac{1}{2}n + 1$, so $n - k = \frac{1}{2}n - 1$. However, in the same way as before, we see that for $n > 6$ no divisor of n is equal to $\frac{1}{2}n - 1$, as the next largest divisor after $\frac{1}{2}n$ is at most $\frac{1}{3}n < \frac{1}{2}n - 1$. This contradicts $n - k$ being a divisor of n.

In the former case, we have $1 - k = -n$, so $k = n + 1$. As n is composite, there exists a divisor d with $1 < d < n$. Then $d - k = d - n - 1$ is also a divisor of n. However, $d \leq \frac{1}{2}n$, so $d - n - 1 \leq -\frac{1}{2}n - 1$, but the only divisor that is at most $-\frac{1}{2}n - 1$ is $-n$. We deduce that $d - n - 1 = -n$, so $d = 1$, which is a contradiction.

Therefore the n that have the given property are precisely those n for which either $n \leq 6$ or n is prime. □

3. We consider the configuration in the figure.



Let M be the midpoint of AB, and let N be the midpoint of AC. From what is given in the problem, it follows that E lies on the perpendicular bisector of AB, so $\angle BME = 90°$. Since we also have $\angle BTE = 90°$ and $\angle BDE = 90°$, by Thales's theorem it follows that $BDTEM$ is a cyclic pentagon. In the same way, it follows that $CFDTN$ is a cyclic pentagon. Now $\angle NTM = 360° - \angle DTN - \angle MTD = (180° - \angle DTN) + (180° - \angle MTD) = \angle DCN + \angle MBD = \angle BCA + \angle ABC = 180° - \angle CAB = 180° - \angle NAM$. Therefore $AMTN$ is a cyclic quadrilateral (which also follows directly from Miquel's theorem).

Now let us chase some angles in order to show that $\triangle TBA \sim \triangle TAC$. We have $\angle NTC = \angle NFC = 90° - \angle FCN$ since $\angle FNC = 90°$. Moreover, we have $\angle FCN = \angle FCA = \angle CAF$ since $|AF| = |CF|$, so $\angle NTC = 90° - \angle CAF = 90° - \angle CAD = \angle DCA = \angle BCA$. Therefore $\angle ACT = \angle NCT = 180° - \angle NTC - \angle CNT = 180° - \angle BCA - \angle CNT$. As MN is a mid-parallel in triangle ABC, we have $MN \parallel BC$ and it follows that $180° - \angle BCA = \angle CNM$. Hence $\angle ACT = \angle CNM - \angle CNT = \angle TNM = \angle TAM = \angle TAB$ (using the cyclicity of the quadrilateral $AMTN$). Similarly, we have $\angle ABT = \angle TAC$, so $\triangle TBA \sim \triangle TAC$. It follows that $\frac{|TB|}{|TA|} = \frac{|TA|}{|TC|}$, so $|TA|^2 = |TB| \cdot |TC|$, as required. □

4. Consider a group of students in which all students know each other, and which has the maximal number of students possible. (Such a group exists, as there are only finitely many students in the classroom, and there exists a group of students in which all students know one another, e.g. a group consisting of only one student.) Denote this group of students by X. We show that by dividing the students in X and the group of students not in X, we may achieve a division with the required properties.

Note that it suffices to show that in the group of students not in X, none of the students know each other. Suppose for a contradiction that A and B are students not in X who know each other. As X is chosen to have the maximal number of students possible, there is a student A' in X who doesn't know A, and a student B' who doesn't know B. We first show that we can take A' and B' to be different. If not, then there is a unique student C in X who doesn't know A, and a unique student in X who doesn't know B (and they are the same student). In other words all other students in X know both A and B, so by replacing C by A and B, we obtain a group of students in which all students know one another; all students in X know each other, A and B know everyone in X (aside from C), and A and B know each other. However this group is larger than X, which is a contradiction.

Therefore we may assume that A' and B' are distinct. Note that A' and B' are in X, so they know each other. Now if A, B, B', A', in that order, take seats around a round table, everyone knows precisely one of his neighbours; A and B, and A' and B' know each other, but A and A', and B and B' do not know each other. This contradicts the given property. Therefore dividing the students into X and the group of students not in X gives a division with the required properties. $\square$

IMO Team Selection Test 3, June 2018

Problems

1. A set of lines in the plan is called *nice* if every line in the set intersects an odd number of other lines in the set.
Determine the smallest integer $k \geq 0$ having the following property: for each 2018 distinct lines $\ell_1, \ell_2, \ldots, \ell_{2018}$ in the plane, there exist lines $\ell_{2018+1}, \ell_{2018+2}, \ldots, \ell_{2018+k}$ such that the lines $\ell_1, \ell_2, \ldots, \ell_{2018+k}$ are distinct and form a nice set.

2. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) - f(y^2) \leq \big(f(x) + y\big)\big(x - f(y)\big)$$

for all $x, y \in \mathbb{R}$.

3. Determine all pairs (a, b) of positive integers such that $(a+b)^3 - 2a^3 - 2b^3$ is a power of two.

4. In a non-isosceles triangle ABC the centre of the incircle is denoted by I. The other intersection point of the angle bisector of $\angle BAC$ and the circumcircle of $\triangle ABC$ is D. The line through I perpendicular to AD intersects BC in F. The midpoint of the circle arc BC on which A lies, is denoted by M. The other intersection point of the line MI and the circle through B, I and C, is denoted by N. Prove that FN is tangent to the circle through B, I and C.

Solutions

1. First we prove that the number of lines in a nice set is even. Suppose on the contrary that the number of lines is odd. Then on each of the odd number of lines, there is an odd number of intersection points, so the total number of intersection points is odd. However, each intersection point is counted twice (once for each of the lines on which is is lying), so the total should be even, which is a contradiction. So the total number of lines in a nice set has to be even. In particular, $2018 + k$ must be even, so k must be even.

Now suppose that there are 1009 directions, such that in each direction there are two of the original lines. Then each line is parallel to exactly one other line. If $k < 1010$, then $k \leq 1008$, so there must be a direction in which we do not add a line. Consider one of the original lines ℓ in this direction. Within the final set of all lines, it intersects all lines except itself and the line it is parallel to. This is an even number. Hence, the resulting set is not nice. Therefore, there is an example in which at least 1010 are needed.

We will now show that it is possible to add exactly 1010 lines such that the resulting set is nice. We consider all directions in which there is an even number (greater than 0) of the original lines. There are at most 1009 such directions. For each of these directions, we add one line. Each line is then parallel to an even number (possibly 0) of other lines. First suppose that the total number of lines is even. Then each lines intersects an odd number of lines (the total set minus itself and an even number of other lines). Now suppose that the total number of lines is odd. Then each line intersects an even number of lines. We ass a line in a new direction, which therefore intersects all lines (an odd number), so that each line after that intersects an odd number of other lines.

Now we have a nice set, and we have added at most $1009 + 1 = 1010$ lines. Possibly, there are less than 1010. In this case, the number of lines added is even, because the total number of lines in a nice set is always even. We choose a direction in which there is at least one line. We then keep adding two lines in this direction. These lines intersect all lines in the other direction, which is an odd number in total. These other lines each got two new intersection points, so they still have an odd number of intersection points. This also holds for the existing lines in the chosen direction, because they get zero new intersection points. The set is still nice. We keep adding pairs of lines until we have 1010 lines in total.

We conclude that the minimum k satisfying the conditions is $k = 1010$. $\square$

2. Substituting $x = y = 0$ yields $0 \le f(0) \cdot -f(0)$. However, squares are non-negative, so this yields $f(0)^2 = 0$ and hence $f(0) = 0$. Now taking $x = 0$ and $y = t$, we obtain $-f(t^2) \le t \cdot -f(t)$, while taking $x = t$ and $y = 0$ yields $f(t^2) \le f(t) \cdot t$. We find $tf(t) \le f(t^2) \le tf(t)$, hence $f(t^2) = tf(t)$ for all $t \in \mathbb{R}$. On the left hand side of the function inequality we can replace $f(x^2) - f(y^2)$ by $xf(x) - yf(y)$. By expanding on the right hand side, we obtain $xf(x) - yf(y) + xy - f(x)f(y)$, hence

$$f(x)f(y) \le xy.$$

Then $f(t^2) = tf(t)$ yields $f(1) = -f(-1)$. By first substituting $y = 1$ and then $y = -1$, we obtain, for all $x \in \mathbb{R}$:

$$x \ge f(x)f(1) = -f(x)f(-1) \ge -x \cdot -1 = x.$$

Hence, equality must hold, which means that $f(x)f(1) = x$. In particular, this yields $f(1)^2 = 1$, hence $f(1) = 1$ or $f(1) = -1$. In the former case, we get $f(x) = x$ for all x and in the latter case, we get $f(x) = -x$ for all x.

When we check $f(x) = x$ with the original function inequality, we get $x^2 - y^2$ on the left hand side and $(x + y)(x - y) = x^2 - y^2$ on the right hand side, hence this function satisfies the inequality. When we check $f(x) = -x$, the left hand side becomes $-x^2 + y^2$ and the right hand side becomes $(-x + y)(x + y) = -x^2 + y^2$, so also this function is a solution.

We conclude that the solutions are: $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = -x$ for all $x \in \mathbb{R}$. $\square$

3. First we determine the pairs (a, b) with $\gcd(a, b) = 1$. We have

$$\begin{aligned}
(a+b)^3 - 2a^3 - 2b^3 &= a^3 + 3a^2b + 3ab^2 + b^3 - 2a^3 - 2b^3 \\
&= -a^3 - b^3 + 3ab(a+b) \\
&= -(a+b)(a^2 - ab + b^2) + 3ab(a+b) \\
&= (a+b)(-a^2 + ab - b^2 + 3ab) \\
&= (a+b)(-a^2 - b^2 + 4ab).
\end{aligned}$$

This has to be a power of two. Because $a + b$ is positive, the other factor must also be positive, and they are both powers of two. As $\gcd(a, b) = 1$, at least one of a and b is odd. Then $a^2 + b^2$ is congruent to 1 or 2 modulo 4, hence $-a^2 - b^2 + 4ab$ is congruent to 3 or 2 modulo 4. However, it is also a power of two, so 3 modulo 4 is impossible, and in the case of 2 modulo 4 it has to equal 2. We conclude that $-a^2 - b^2 + 4ab = 2$.

Suppose that $a+b$ is at least 8. Because it is a power of two, it is also divisible by 8, hence we can write $b = 8m - a$ for a certain positive integer m. Then we have

$$\begin{aligned} 2 &= -a^2 - b^2 + 4ab \\ &= -a^2 - (8m-a)^2 + 4a(8m-a) \\ &= -a^2 - 64m^2 + 16ma - a^2 + 32ma - 4a^2 \\ &= -6a^2 + 48ma - 64m^2, \end{aligned}$$

hence

$$1 = -3a^2 + 24ma - 32m^2.$$

Modulo 8, this becomes $1 \equiv -3a^2$. If a is even, then the right hand side is even, and if a is odd, then the right hand side is -3 modulo 8. In both cases, the equality does not hold modulo 8. We conclude that $a+b$ is not greater or equal to 8. Because a and b are both at least 1, $a+b$ is equal to 2 or 4.

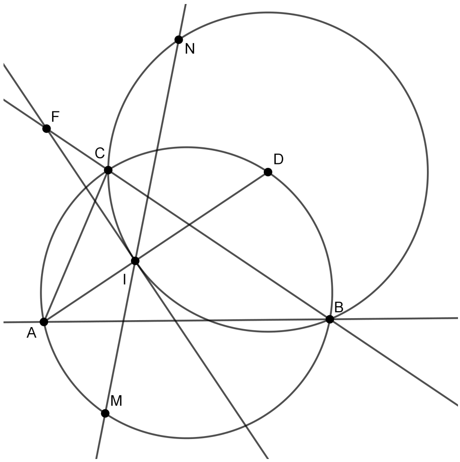
- If $a+b=2$, then $a=b=1$. Now we have $-a^2-b^2+4ab=2$, hence the product of the two factors is a power of two (namely 4). Hence $(1,1)$ is a solution.
- If $a+b=4$, then because of $\gcd(a,b)=1$ we have either $a=1$ and $b=3$ or the other way around. Now $-a^2-b^2+4ab=2$, so also the product of the two factors is a power of two (namely 8). Hence $(1,3)$ and $(3,1)$ are solutions.

Now consider a solution (a,b) with $\gcd(a,b)=d>1$. Then d is a divisor of $(a+b)^3-2a^3-2b^3$, hence d has to be a power of two itself. We now remark that $(\frac{a}{d},\frac{b}{d})$ is another solution, because d^3 is divided out from the power of two $(a+b)^3-2a^3-2b^3$, leaving another power of two. Hence, $(\frac{a}{d},\frac{b}{d})$ has to be one of the three previously found pairs. On the other hand, each of these three pairs multiplied by an arbitrary power of two is another solution.

We conclude that all possible solutions are given by

$$(a,b)=(2^k,2^k), \quad (a,b)=(2^k,3\cdot 2^k), \quad \text{and} \quad (a,b)=(3\cdot 2^k,2^k),$$

where k runs through all non-negative integers. □



4. We consider the configuration in which $|AB| > |AC|$, causing C to lie between B and F. For other configurations, the proof is analogous.

Because $\angle DAC = \angle BAD$ holds, D is the midpoint of the circle arc BC on which A does not lie. Hence, the line DM is a diameter of the circle. Thales's theorem yields $\angle DBM = 90°$. Let K be the intersection point of BC and DM. As DM is the segment bisector of BC, we have $\angle BKM = 90° = \angle DBM$. Therefore, $\triangle MKB \sim \triangle MBD$ (AA). This yields $\frac{|MB|}{|MK|} = \frac{|MD|}{|MB|}$, hence $|MD| \cdot |MK| = |MB|^2$. Because K is on the interior of MD, this equality also holds using directed distances: $MD \cdot MK = MB^2$.

Because D is the midpoint of arc BC, we have $|DB| = |DC|$. Moreover, $\angle CDI = \angle CDA = \angle CBA$ and $\angle DCI = \angle DCB + \angle BCI = \angle DAB + \angle BCI = \frac{1}{2}\angle CAB + \frac{1}{2}\angle BCA$. Using the sum of the angles in triangle DCI, we now get that $\angle DIC = 180° - \angle CBA - \frac{1}{2}\angle CAB - \frac{1}{2}\angle BCA = \frac{1}{2}\angle CAB + \frac{1}{2}\angle BCA = \angle DCI$. Hence, triangle DCI is isosceles with $|DC| = |DI|$. We conclude that D is the centre of the circle through B, C and I. Now DB is the radius of this circle, and DB is perpendicular to MB, which yields that MB is a tangent line. Using the secant-tangent theorem, we get $MB^2 = MI \cdot MN$. Together with the previous paragraph we conclude that $MD \cdot MK = MI \cdot MN$. Hence, $NDKI$ is a cyclic quadrilateral because of the same theorem.

We also know that $\angle FID = 90° = \angle FKD$, hence $DKIF$ is a cyclic quadrilateral. We conclude that $NDKIF$ is a cyclic pentagon. This yields $\angle DNF = 180° - \angle DIF = 90°$. Hence, NF is perpendicular to the radius DN of the circle through B, I and C; this means that FN is tangent to this circle. $\square$

Junior Mathematical Olympiad, September 2017

Problems

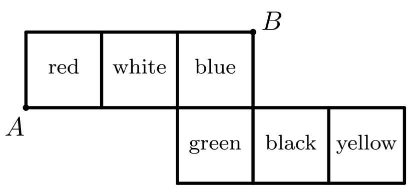
Part 1

1. A positive three-digit number is called *nice* if the sum of the last two digits equals the first digit. For example, 123 is *not* nice, because 1 is not equal to $2+3$. How many three-digit numbers are nice?
Note that a three-digit number cannot start with digit 0*.*

A) 45 B) 48 C) 50 D) 54 E) 55

2. The faces of a cube have different colours. In the figure you can see a net for the cube. The points A and B in the net correspond to two vertices of the same face of the cube.
What colour does that face have?

A) red B) blue C) green D) black E) yellow



3. We consider sequences of 20 integers. The integers can be positive or negative, but cannot be zero. Also, we impose the following conditions on the sequences: of any two adjacent numbers at least one is positive; the sum of any three adjacent numbers is negative; the product of any four adjacent numbers is positive.
Consider the following four statements about such sequences:

- There can never be two adjacent positive numbers.
- There may be more positive than negative numbers.
- The sum of all 20 numbers is always negative.
- The number -1 can never occur.

How many of these statements are true?

A) 0 B) 1 C) 2 D) 3 E) 4

4. The number $n^2 + 21$ is the square of an integer.
For how many positive integers n does this hold?

A) 0 B) 1 C) 2 D) 3 E) 4

5. Sanne is building a $9 \times 9 \times 9$ cube by gluing $1 \times 1 \times 1$ blocks together. She doesn't have quite enough blocks to complete the task. Therefore, she decides to leave out some of the blocks from the large cube. In order to still get a nice rigid cube, she makes sure that no two holes (left out blocks) are adjacent. In fact, two holes should not even touch in an edge or a single vertex. Also, she does not leave out any of the blocks on the outside of the cube.
What is the minimum number of blocks that Sanne needs to build the cube?

A) 365 B) 604 C) 665 D) 673 E) 702

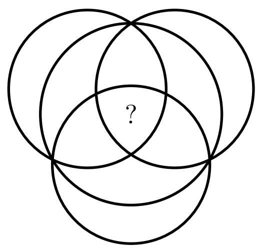
6. Peter starts out with the numbers 1, 2, 3, and 4. He may take two of his numbers and replace them by their sum, their product, or their difference. He performs this replacement step three times, after which a single number remains.

Example. He could replace the 2 and the 4 by $2 + 4 = 6$, then replace the 1 and the 3 by $3 - 1 = 2$, and finally replace the 6 and the 2 by $8 = 2 + 6$. Then, the remaining number would be 8.

Which of the following five numbers *cannot* be the number that remains?

A) 28 B) 30 C) 32 D) 34 E) 36

7. Four circles together enclose ten regions in the plane, as in the figure. We want to place the numbers 1 to 10 inside the regions (one number per region). This must be done in such a way that adding the numbers inside a circle gives the same answer for all four circles.
Which number should be placed in the region with the question mark?



A) 1 B) 2 C) 4 D) 6 E) 7

8. You have a collection of hats. Each hat has three attributes: the colour (red or blue), the shape (top hat or pointed hat), and the pattern (spots or stripes). You put a number of gnomes in a room and put a hat on each of them. For any two gnomes their hats must be different, yet share at least one attribute (for example: both hats are blue). The gnomes can see everyone's hat, except their own. The gnomes are not allowed to communicate with one another.
What is the minimum number of gnomes you have to put in the room in order to be be sure that one of them can determine one of the attributes of his own hat?

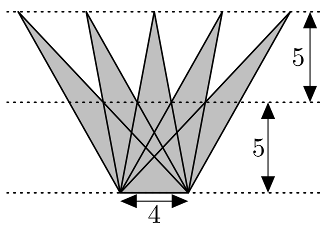
A) 3 B) 4 C) 5
D) 8 E) That is impossible for any number of gnomes.

Part 2

1. Stef has 18 coins of which 17 are identical, but one is slightly lighter than the other coins. Together, the 18 coins weigh 214 grams. Stef removes two of the coins and weighs the remaining 16 coins. Together the 16 coins weigh 190 grams.
How much does the lighter coin weigh?

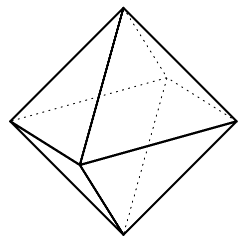
2. We say that a positive integer is *balanced* if the average of the first two digits is 2, the average of the first three digits (if they all exist) is 3, the average of the first four digits (if they all exist) is 4, et cetera.
What is the largest balanced number?

3. What is the area of the crown-shaped area?
Note that the figure is not drawn to scale.



4. A clock has the numbers 1 to 12 for indicating the hours. Ernie has interchanged these twelve numbers in such a way that any two adjacent numbers on the clock differ by either 2 or 3. Fortunately, the number 12 is still in the right place, but the 9 is where the 1 is supposed to be.
What number is in the place where the 9 is supposed to be?

5. The numbers 1 to 8 are assigned to the eight faces of an octahedron. For each vertex, we compute the sum of the four numbers on the faces meeting in that vertex. For four of the vertices we get the same outcome. For a fifth vertex the outcome is 16.
What is the outcome for the sixth vertex?



6. When a, b, c, and d are digits, we denote by $\overline{abcd}$ the number composed of those four digits. The numbers $\overline{abcd}$ and $\overline{cbad}$ are both perfect squares. The number $\overline{bad}$ is the cube of a positive integer.
Determine the number $\overline{abcd}$.

7. An employee at the supermarket is stacking crisps canisters. There are two types of canisters: small ones and large ones. Three small canisters stack to the same height as one large canister. The employee makes a stack of 12 small canisters. Next to it, he makes more stacks of the same height, but all stacks are different. (If one stack starts with a small canister followed by a large one, and the other starts with a large one followed by a small one, the two stacks are different.)
How many different stacks can he make, including the first stack?

8. You may choose any number consisting of five different digits. The digit 0 *cannot* be used. The next step is to choose two adjacent digits and switch their positions. You may perform this step five times in total. Finally, you compute the difference between the initially chosen number and the final number obtained after switching.
What is the largest possible difference that can be obtained?

Example. Suppose you choose the number 47632. Then you could switch digits to obtain, in this order, the numbers 46732, 46372, 46327, 43627, and 34627. Then, the difference between the initial number and the final number is $47632 - 34627 = 13005$.

Answers

Part 1

1. D) 54

2. D) black

3. C) 2 (statements 1 and 3)

4. C) 2

5. C) 665

6. D) 34

7. E) 7

8. E) That is impossible for any number of gnomes.

Part 2

1.	10 grams	**5.**	20
2.	40579	**6.**	1296
3.	60	**7.**	60
4.	5	**8.**	85230

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