54th Dutch Mathematical Olympiad 2015

Contents

Introduction

The selection process for IMO 2016 started with the first round in January 2015, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In this first round 10277 students from 354 secondary schools participated.

The 1000 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 130 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 54rd edition 2015.

The 32 most outstanding candidates of the Dutch Mathematical Olympiad 2015 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

In February a team of four girls was chosen from the training group to represent the Netherlands at the EGMO in Buşteni, Romania, from 10 until 16 April. The team brought home a silver medal and two bronze medals, a very nice achievement. For more information about the EGMO (including the 2016 paper), see www.egmo.org.

In March a selection test of three and a half hours was held to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Soest, the Netherlands, from 29 April until 1 May. The Dutch team received six bronze medals and three silver medals, and managed to get the highest total score. For more information about the BxMO (including the 2016 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2016 was selected by three team selection tests on 2, 3 and 4 June 2016, each lasting four hours. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Hong Kong, from 30 June until 9 July.

For younger students the Junior Mathematical Olympiad was held in October 2015 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2016 in Hong Kong consists of

- Erik van Cappellen (17 years old)
  - participated in BxMO 2016
- Wietze Koops (15 years old)
  - bronze medal at BxMO 2016
- Levi van de Pol (14 years old)
  - silver medal at BxMO 2015, silver medal at BxMO 2016
  - observer C at IMO 2015
- Reinier Schmiermann (14 years old)
  - silver medal at BxMO 2016
- Pim Spelier (16 years old)
  - silver medal at BxMO 2016
- Gabriel Visser (18 years old)
  - bronze medal at BxMO 2016

We bring as observer C the promising young student

- Matthijs van der Poel (15 years old)
  - bronze medal at BxMO 2016

The team is coached by

- Julian Lyczak (team leader), Leiden University
- Birgit van Dalen (deputy leader), Leiden University
- Merlijn Staps (observer B), Utrecht University

First Round, January 2015

Problems

A-problems

1. A square is divided into two rectangular pieces by a straight line. The sum of the circumferences of the two rectangles is 30 centimetres.
What is the side length of the square in centimetres?

A) 5 B) 6 C) $\frac{15}{2}$ D) 8 E) 12

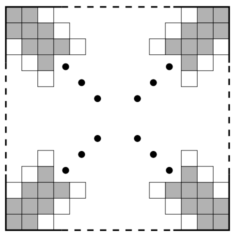
2. Five suspects are being questioned about the order of arrival at a crime scene. They make the following statements.

Aad: "I arrived first."
Bas: "I arrived second."
Carl: "I arrived third."
Dave: "Of Aad and Bas, one arrived before me and the other after me."
Erik: "Of Bas and Carl, one arrived before me and the other after me."

It is known that exactly one of the suspects lied.
Who was the fourth to arrive at the crime scene?

A) Aad B) Bas C) Carl D) Dave E) Erik

3. A big square consists of 2015 times 2015 small squares. The small squares on the two main diagonals and on the four adjacent diagonals are coloured grey, and the rest is coloured white (see the figure).
How many small squares are coloured grey?

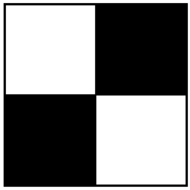


A) 12081 B) 12082 C) 12085
D) 12086 E) 12090

4. The difference of two integers is 10. If you multiply the two integers, you will get one of the following five numbers.
Which number do you get?

A) 22398 B) 22420 C) 22442 D) 22453 E) 22475

5. Jan has got a wooden cube. He divides each of the faces into 2×2 squares that he subsequently paints in a black-white pattern: two diagonally opposite squares are painted black, the other two are painted white. In each vertex of the cube three squares meet. If two or three of these squares are black, we call the vertex a *dark* vertex.



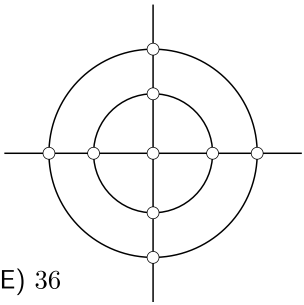
What is the smallest number of dark vertices that the cube can have?

A) 0 B) 1 C) 2 D) 3 E) 4

6. In how many ways can you get the number 100 by adding some consecutive integers between 1 and 99 inclusive?

A) 1 B) 2 C) 3 D) 4 E) 5

7. In the figure, you see two circles and two lines together with the nine nodes in which they intersect. Jaap wants to colour exactly four of the nodes red, in such a way that no three red nodes are on the same line or on the same circle.

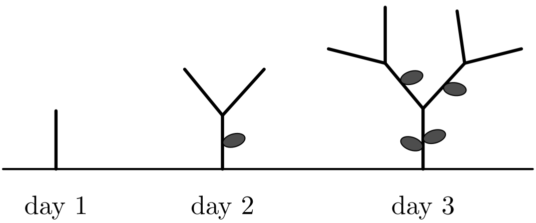


How many such colourings can Jaap make?

A) 6 B) 12 C) 18 D) 24 E) 36

8. A tree grows in the following manner. On day 1, one branch grows out of the ground. On day 2, a leaf grows on the branch and the branch tip splits up into two new branches. On each subsequent day, a new leaf grows on every existing branch and each branch tip splits up into two new branches. See the figure below.
How many leaves does the tree have at the end of the tenth day?

A) 172 B) 503 C) 920 D) 1013 E) 2047

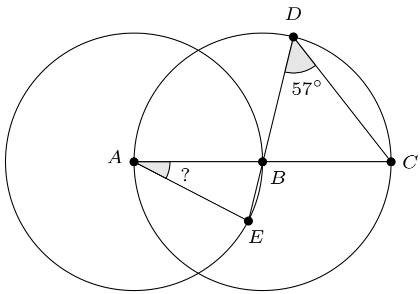


B-problems

The answer to each B-problem is a number.

1. Julia constructs a sequence of numbers. She starts with two integers she chooses herself. Then, she calculates the next numbers in the sequence as follows: if the last number she wrote down is b and the number before that is a, then the next number will be $2b - a$. The second number in Julia's sequence is 55 and the hundredth number is 2015.
What is the first number in Julia's sequence?

2. Two points A and B and two circles are given, one having A as centre and going through B and the other one having B as centre and going through A. Point C lies on the second circle and on line AB. Point D also lies on the second circle. Point E lies on the first circle and on line BD. See the figure below. Moreover, $\angle D = 57°$.
What is the value of $\angle A$ in degrees?



3. A positive integer is called *alternating* if its digits alternate between even and odd. For example, 2381 and 3218 are alternating, but 2318 is not. An integer is called *super alternating* if the number itself is alternating and twice that number is alternating as well. For example, 505 is super alternating, because both 505 and 1010 are alternating.
How many super alternating integers consisting of four digits exist?
Pay attention: a four digit integer cannot start with a 0.

4. On a school trip, twenty students will be abseiling. In each round, one student will descend the mountain. Hence, after twenty rounds, all students will have gone down the mountain safely. In the first round, cards bearing the numbers 1 to 20 are distributed among the students. The student getting number 1 will go down first. In round 2, cards bearing the numbers 1 to 19 are distributed among the remaining students. The student receiving the number 1 is next to descend. They continue in this way, until there is only one student left in round 20, who automatically gets a card bearing the number 1. By an amazing coincidence, no student gets the same number twice. In the first round, Sara gets a card with number 11.
What is the sum of the numbers on the cards received by Sara?

Solutions

A-problems

1. A) 5

2. E) Erik

3. A) 12081

4. E) 22475

5. C) 2

6. B) 2

7. C) 18

8. D) 1013

B-problems

1. 35

2. $48°$

3. 70

4. 66

Second Round, March 2015

Problems

B-problems

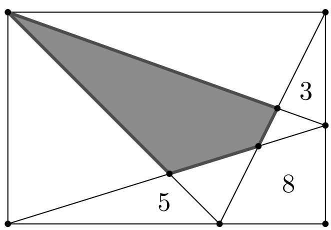
The answer to each B-problem is a number.

1. We consider numbers consisting of two or more digits with no digit being 0. Such a number is called *thirteenish* if every two consecutive digits form a multiple of 13. For example: 139 is thirteenish because $13 = 1 \times 13$ and $39 = 3 \times 13$.
How many five digit numbers are thirteenish?

2. A quadrilateral $ABCD$ has right angles at A and B. Also, $|AB| = 5$ and $|AD| = |CD| = 6$.
Determine all possible values of $|BC|$.

3. Berry has picked 756 raspberries. He divides the raspberries among himself and his friends in such a way that everyone gets the same number of raspberries. However, three of his friends are not feeling hungry and they each return a number of raspberries: exactly one quarter of their share. Berry has a healthy appetite and eats not only his own share, but the returned raspberries as well. Berry has lost count, but does know for a fact that he has eaten more than 150 raspberries.
How many raspberries did Berry eat?

4. Four line segments divide a rectangle into eight pieces as indicated in the figure. For three of the pieces, the area is indicated as well: 3, 5, and 8.



What is the area of the grey quadrilateral?

5. In the cells of a 5×5-table, the numbers 1 to 5 are placed in such a way that in every row and in every column, each of the five numbers occurs exactly once. A number in a given row and column is *well-placed* if the following conditions are met.

- In that row, all smaller numbers are to the left of the number and all larger numbers are to the right of it, or conversely.
- In that column, all smaller numbers are below the number and all larger numbers are above it, or conversely.

What is the maximum number of well-placed numbers in such a table?

C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

1. A set of different numbers are *evenly spread* if after sorting them from small to large, all pairs of consecutive numbers have the same difference. For example: 3, 11 and 7 are evenly spread, because after sorting them, both differences are 4.

a) Kees starts out with three different numbers. He adds each pair of these numbers to obtain three outcomes. According to Jan, these three outcomes can be evenly spread only if the three starting numbers were evenly spread.
Is Jan right? If so, prove this; if not, use an example to prove that Jan is wrong.

b) Jan starts out with four different numbers. He also adds each pair of them to obtain six outcomes. He wants to choose his four numbers in such a way that the six resulting numbers are evenly spread.
Is this possible? If so, give an example; if not, prove that it is impossible.

2. We consider rectangular boards consisting of $m \times n$ cells that are arranged in m (horizontal) rows and n (vertical) columns. We want to colour each cell of the board black or white in such a way that the following rules are obeyed.

- For every row, the number of white cells equals the number of black cells.
- If a row and a column meet in a *black* cell, the row and column contain equal numbers of black cells.
- If a row and a column meet in a *white* cell, the row and column contain equal numbers of white cells.

Determine all pairs (m, n) for which such a colouring is possible.

Solutions

B-problems

1. 6

2. $6 \pm \sqrt{11}$

3. 189

4. 16

5. 5

C-problems

1. a) Kees starts with three numbers $a < b < c$. The three sums are then ordered as follows: $a + b < a + c < b + c$. If these are evenly spread, then the difference $(b+c)-(a+c) = b-a$ equals $(a+c)-(a+b) = c-b$. This is exactly the condition for the three original numbers a, b and c to be evenly spread. Hence, Jan was right.

b) Jan can accomplish this by taking the four numbers $0, 1, 2$ and 4 to start with. The six results then are $0 + 1 = 1$, $0 + 2 = 2$, $1 + 2 = 3$, $0 + 4 = 4$, $1 + 4 = 5$, and $2 + 4 = 6$. These six numbers are evenly spread. $\square$

2. In the following cases, a colouring meeting all the requirements exists.

- $m = n$ is even
  We colour the board as in a chessboard pattern. That is: in each row and column the squares are alternately black and white. This colouring meets all requirements.
- $n = 2m$
  We colour all the squares in the left half of the board white, and colour all the squares in the right half of the board black. This colouring meets all requirements.

Now we shall show that these are the only possible board sizes. Consider a coloured board that meets all requirements. Because each row has equally many black and white squares, the total number of squares in a row must be divisible by 2. Write $n = 2k$. Each row has exactly k white and k black squares. Now consider the left column. If all its squares are white, then the column has k white squares because of the second requirement. Hence, we have $m = k$ in this case. The same happens when all squares in the left column are black. If there are both black and white squares in the left column, then there must be exacly k white and k black squares because of the second requirement. Hence, we find $m = 2k = n$ in this case.

We conclude that for a pair (m, n) there exists a colouring if and only if $n = 2m$ or if $m = n$ and n is even. $\square$

Final Round, September 2015

Problems

1. We make groups of numbers. Each group consists of five distinct numbers. A number may occur in multiple groups. For any two groups, there are exactly four numbers that occur in both groups.

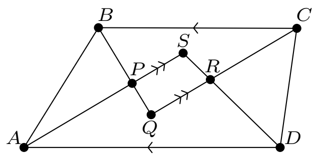
(a) Determine whether it is possible to make 2015 groups.

(b) If all groups together must contain exactly *six* distinct numbers, what is the greatest number of groups that you can make?

(c) If all groups together must contain exactly *seven* distinct numbers, what is the greatest number of groups that you can make?

2. On a 1000×1000-board we put dominoes, in such a way that each domino covers exactly two squares on the board. Moreover, two dominoes are not allowed to be adjacent, but are allowed to touch in a vertex.
Determine the maximum number of dominoes that we can put on the board in this way.
Attention: you have to really prove *that a greater number of dominoes is impossible.*

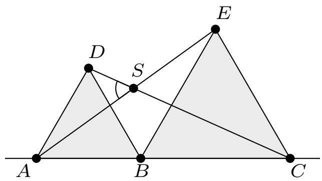
3. Version for junior students



In quadrilateral $ABCD$ sides BC and AD are parallel. In each of the four vertices we draw an angular bisector. The angular bisectors of angles A and B intersect in point P, those of angles B and C intersect in point Q, those of angles C and D intersect in point R, and those of angles D and A intersect in point S. Suppose that PS is parallel to QR.
Prove that $|AB| = |CD|$.
Attention: the figure is not drawn to scale.

3. Version for senior students

Points A, B, and C are on a line in this order. Points D and E lie on the same side of this line, in such a way that triangles ABD and BCE are equilateral. The segments AE and CD intersect in point S.
Prove that $\angle ASD = 60°$.



4. Find all pairs of prime numbers (p, q) for which

$$7pq^2 + p = q^3 + 43p^3 + 1.$$

5. Given are (not necessarily positive) real numbers a, b, and c for which

$$|a-b| \geq |c|, \qquad |b-c| \geq |a|, \quad \text{and} \quad |c-a| \geq |b|.$$

Here $|x|$ is the absolute value of x, i.e. $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.
Prove that one of the numbers a, b, and c is the sum of the other two.

Solutions

1. (a) It is possible to make 2015 groups. For example, take the 2015 groups $\{-4, -3, -2, -1, i\}$, where i runs from 1 to 2015. Each group consists of five distinct numbers, as required, and any two groups have exactly four numbers in common: -4, -3, -2, and -1.

(b) Using six available numbers, there are only six possible groups of five numbers (each obtained by leaving out one of the six numbers). Those six groups do satisfy the requirement that any two of them have exactly four numbers in common. We conclude that six is the greatest number of groups we can make in this case.

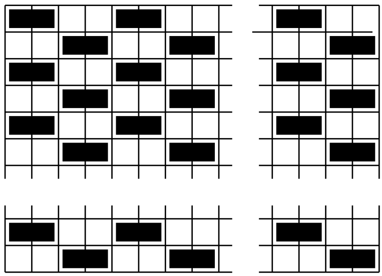
(c) A way to make three groups is to take $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 6\}$, and $\{1, 2, 3, 4, 7\}$.

More than three groups is not possible. Indeed, suppose we have four or more groups. The first two groups are $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, f\}$, where a, b, c, d, e, and f are distinct numbers. Then there must be a third group C containing a seventh number g. The remaining four numbers in C must be in both A and B, hence $C = \{a, b, c, d, g\}$.

Now consider a hypothetical fourth group D. This group cannot contain the number g since otherwise, using a similar reasoning as for C, we would have $D = \{a, b, c, d, g\}$. Because D does not contain the number g, it must contain the remaining four numbers a, b, c, and d from C. Comparison with groups A and B then shows that D can contain neither e nor f. It follows that besides a, b, c, and d, D cannot contain a fifth number, contradicting the requirements.

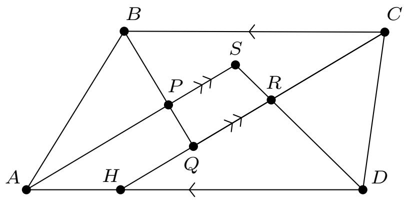
We conclude that the greatest number of groups we can make is three.

2. A maximum of 250,000 dominoes can fit on the board. We first show to place this number of dominoes on the board. In each row we put 250 dominoes with two empty squares in between consecutive dominoes. In the odd numbered rows we start with a domino and end with two empty squares (since the number of squares in a row is a multiple of four). In the even numbered rows we start with two empty squares and end with a domino. Thus, we place a total of $1000 \cdot 250 = 250{,}000$ dominoes, see the figure. Clearly, no two dominoes in the same row are adjacent, and dominoes in adjacent rows touch in a vertex, at most.



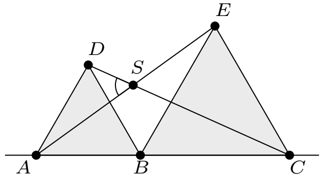
To complete the proof, we need to show that it is not possible to place more than 250,000 dominoes on the board. Partition the board into 500×500 patches consisting of 2×2 squares each. Of each patch, at most two out of the four squares can be covered by dominoes since otherwise two dominoes would be adjacent. Hence, no more than $2 \cdot 500 \cdot 500 = 500{,}000$ squares can be covered by dominoes. This shows that no more than 250,000 dominoes can fit on the board.

3. Version for junior students



The intersection of CQ and AD is called H. We have $\frac{1}{2}\angle BAD = \angle SAD = \angle CHD$ (corresponding angles). Also, we have $\angle CHD = \angle HCB = \frac{1}{2}\angle DCB$ (alternate interior angles). It follows that the two angles $\angle BAD$ and $\angle DCB$ of quadrilateral $ABCD$ are equal. This implies that $\angle BAD + \angle ADC = \angle DCB + \angle ADC = 180°$, because AD and BC are parallel. From $\angle BAD + \angle ADC = 180°$ it follows that AB and CD are parallel. Hence, $ABCD$ is a parallelogram. We conclude that $|AB| = |CD|$.

3. Version for senior students



Observe that $\angle ABE = 180° - \angle EBC = 120°$ and $\angle DBC = 180° - \angle ABD = 120°$. Furthermore, $|AB| = |DB|$ and $|BE| = |BC|$. It follows that triangles ABE and DBC are congruent (SAS). In particular, $\angle EAB = \angle CDB$.

Observe that $\angle ASD = 180° - \angle SDA - \angle DAS = 180° - (60° + \angle CDB) - \angle DAE$. Substituting $\angle CDB = \angle EAB$ shows that $\angle ASD = 120° - \angle EAB - \angle DAE = 120° - 60° = 60°$.

4. We start by observing that in the equation $7pq^2 + p = q^3 + 43p^3 + 1$ the numbers p and q cannot both be odd. Otherwise, $7pq^2 + p$ would be even, while $q^3 + 43p^3 + 1$ would be odd. Since 2 is the only even prime number, we conclude that $p = 2$ or $q = 2$.

In the case $p = 2$, we obtain the equation $14q^2 + 2 = q^3 + 344 + 1$, which can be rewritten as $q^3 - 14q^2 = -343$. This shows that q must be a divisor of $343 = 7^3$, hence $q = 7$. Substitution confirms that $(p, q) = (2, 7)$ is indeed a solution since $14q^2 + 2 = 2 \cdot 7 \cdot 7^2 + 2$ and $q^3 + 344 + 1 = 7^3 + (7^3 + 1) + 1 = 2 \cdot 7^3 + 2$ are equal.

Next, we consider the case that $q = 2$ and p is odd. This results in the equation $28p + p = 8 + 43p^3 + 1$. Since p is odd, we see that $28p + p$ is odd, while $8 + 43p^3 + 1$ is even. Hence, no solutions exist in this case.

We conclude that $(p, q) = (2, 7)$ is the only solution.

5. The system of inequalities is symmetric in the variables a, b, and c: if we exchange two of these variables, the system remains unchanged (up to rewriting it). For example, if we exchange variables a and b, we obtain

$$|b - a| \geq |c|, \quad |a - c| \geq |b|, \quad |c - b| \geq |a|.$$

Since $|b - a| = |a - b|$, $|a - c| = |c - a|$, and $|c - b| = |b - c|$, this can be rewritten as

$$|a - b| \geq |c|, \quad |c - a| \geq |b|, \quad |b - c| \geq |a|,$$

obtaining the original system of inequalities. Due to this symmetry, we may assume without loss of generality that $a \geq b \geq c$.

First observe that $c \leq 0$. Indeed, if $a \geq b \geq c > 0$ would hold, then $|b - c| \geq |a|$ would imply that $b - c \geq a$. Rewriting gives $b \geq a + c > a$, which contradicts the fact that $b \leq a$.

Next, consider the following series of inequalities.

$$|a| + |c| = |a| - c \geq a - c = (a - b) + (b - c) = |a - b| + |b - c| \geq |c| + |a|.$$

Since the first and the last term are equal, we can conclude that all of the above inequalities must be equalities. In particular, $a - b = |a - b| = |c|$. Since $c \leq 0$, this implies that $a + c = b$. This shows that one of the three numbers equals the sum of the other two.

BxMO Team Selection Test, March 2016

Problems

1. For a positive integer n that is not a power of two, we define $t(n)$ as the greatest odd divisor of n and $r(n)$ as the smallest positive odd divisor of n unequal to 1. Determine all positive integers n that are not a power of two and for which we have

$$n = 3t(n) + 5r(n).$$

2. Determine all triples (x, y, z) of non-positive real numbers that satisfy the following system of equations

$$\begin{aligned} x^2 - y &= (z-1)^2, \\ y^2 - z &= (x-1)^2, \\ z^2 - x &= (y-1)^2. \end{aligned}$$

3. Let $\triangle ABC$ be a right-angled triangle with $\angle A = 90^\circ$ and circumcircle Γ. The inscribed circle is tangent to BC in point D. Let E be the midpoint of the arc AB of Γ not containing C and let F be the midpoint of the arc AC of Γ not containing B.

(a) Prove that $\triangle ABC \sim \triangle DEF$.

(b) Prove that EF goes through the points of tangency of the incircle to AB and AC.

4. The Facebook group Olympiad training has at least five members. There is a certain integer k with following property: for each k-tuple of members there is at least one member of this k-tuple friends with each of the other $k-1$. (Friendship is mutual: if A is friends with B, then also B is friends with A.)

(a) Suppose $k = 4$. Can you say with certainty that the Facebook group has a member that is friends with each of the other members?

(b) Suppose $k = 5$. Can you say with certainty that the Facebook group has a member that is friends with each of the other members?

5. Determine all pairs (m, n) of positive integers for which

$$(m+n)^3 \mid 2n(3m^2 + n^2) + 8.$$

Solutions

1. Let p be the smallest odd prime divisor of n. Then $r(n) = p$ holds. Now we can write $n = 2^t mp$ with m odd and $t \geq 0$. Then we have $t(n) = pm$, hence the given equality becomes $2^t mp = 3pm + 5p$, or $(2^t - 3)mp = 5p$, hence $(2^t - 3)m = 5$. We see that m must be a divisor of 5, hence $m = 1$ or $m = 5$. If $m = 1$ holds, then $2^t = 8$, hence $t = 3$. We get that $n = 8p$ with p an odd prime number. This is indeed a solution for all odd prime numbers p. If $m = 5$ holds, then $2^t = 4$, hence $t = 2$. We get that $n = 4 \cdot 5 \cdot p$ with p an odd prime. This only gives a solution if p is the smallest odd prime divisor, which is if $p = 3$ or $p = 5$. In this way, we find two more solutions: $n = 60$ and $n = 100$. $\square$

2. Expanding the right hand sides and adding up all equations gives

$$x^2 + y^2 + z^2 - (x + y + z) = x^2 + y^2 + z^2 - 2(x + y + z) + 3,$$

hence $x + y + z = 3$. Without loss of generality we assume that $x \leq y, z$. Then $0 \leq x \leq 1$ holds. Therefore, $x^2 \leq x$, hence $x^2 - y \leq x - y \leq 0$. On the other hand we have $x^2 - y = (z - 1)^2 \geq 0$. Thus, equality has to hold in $x^2 \leq x$ and $x - y \leq 0$. From the first equality we obtain $x = 0$ or $x = 1$ and the second one yields $x = y$. Suppose $x = y = 0$, then $x + y + z = 3$ implies that $z = 3$. But then we do not have $x^2 - y = (z - 1)^2$, which is a contradiction. We are only left with the case $x = y = 1$. Then we have $z = 3 - 1 - 1 = 1$. This triple indeed satisfies all equations and hence it is the only solution. $\square$

3. (a) The midpoint E of the arc AB not containing C, lies on the angular bisector CI. In the same way, F lies on BI. We have $\angle IFC = \angle BFC = \angle BAC = 90°$, because $ABCF$ is a cyclic quadrilateral and $\angle IDC = 90°$ because D is the point of tangency of the incircle to BC. Hence, $\angle IFC + \angle IDC = 180°$, which yields that $FIDC$ is a cyclic quadrilateral. Now we have $\angle DFI = \angle DCI = \frac{1}{2}\angle ACB$, while also $\angle IFE = \angle BFE = \angle BCE = \frac{1}{2}\angle ACB$. Hence, $\angle DFE = \frac{1}{2}\angle ACB + \frac{1}{2}\angle ACB = \angle ACB$. Analogously, we have $\angle DEF = \angle ABC$. Altogether, this yields $\triangle DEF \sim \triangle ABC$.

(b) Let S be the intersection of EF and AB. In the previous part we already saw that BF is the angular bisector of $\angle DFE = \angle DFS$. Hence, BF is also the angular bisector of $\angle ABC = \angle SBD$. Therefore, $\triangle BDF \cong \triangle BSF$ because of (ASA). This means that $|BD| = |BS|$. On the other hand, the distances of B to the points of tangency of the incircle to BC and BA are equal and one of these points of tangency is D, hence the other point of tangency must be S. Hence, EF goes through the point of tangency of the incircle to AB. Analogously, it also goes through the point of tangency of the incircle to AC. □

4. (a) Yes, you can. If everybody is friends with everyone else, then we are done. Hence, suppose that there are two members, say A and B, who are not friends with each other. If we consider a group of four with A, B, and two other members, then one of the other two must be friends with the other and with A and B. In particular, the two others are friends with each other. This holds for any two members (unequal to A and B) that we choose, hence each pair not containing A and B is friends with each other. Now take A, B, C, and D and suppose that C is friends with A, B, and D. Moreover, he is also friends with all other members of the group, hence C is someone who is friends with everybody else.

(b) No, this is not possible. We give a counterexample. Suppose that the Facebook group has six members, called A, B, C, D, E, and F. They are all friends with each other, except for the pair (A, B), the pair (C, D) and the pair (E, F). This means that no member is friends with every other member. If we take a group of five then without loss of generality this is A, B, C, D, and E. Here we can find someone who is friends with the other four, namely E. Hence, the condition is met. □

5. Suppose that the quotient of $2n(3m^2+n^2)+8$ and $(m+n)^3$ is unequal to 1. Then it is at least 2, hence we have

$$(m+n)^3 \leq n(3m^2+n^2)+4,$$

or

$$m^3+3m^2n+3mn^2+n^3 \leq 3m^2n+n^3+4,$$

or

$$m^3+3mn^2 \leq 4.$$

This yields $m<2$, and therefore $m=1$. Then we have $1+3n^2 \leq 4$, and hence also $n=1$. The pair $(m,n)=(1,1)$ is indeed a solution, because $2^3 \mid 2\cdot 4+8$ holds.

The other possibility is that the quotient does equal 1. Then we have

$$(m+n)^3 = 2n(3m^2+n^2)+8,$$

or

$$m^3+3m^2n+3mn^2+n^3 = 6m^2n+2n^3+8,$$

or

$$m^3-3m^2n+3mn^2-n^3 = 8.$$

The left hand side we can factor as $(m-n)^3$. Hence, we have $m-n=2$, or $m=n+2$. From the previous calculations it also follows that $(m,n)=(n+2,n)$ is indeed a solution for all positive integers n.

We conclude that the solutions are: $(m,n)=(1,1)$ and $(m,n)=(n+2,n)$ for $n\geq 1$. □

IMO Team Selection Test 1, June 2016

Problems

1. Let ABC be an acute triangle. Let H be the foot on AB of the altitude through C. Suppose that $|AH| = 3|BH|$. Let M and N be the midpoints of the segments AB and AC, respectively. Let P be a point such that $|NP| = |NC|$ and $|CP| = |CB|$ and such that B and P lie on opposite sides of the line AC.

Show that $\angle APM = \angle PBA$.

2. Let n be a positive integer, and consider a square of dimensions $2^n \times 2^n$. We cover this square by a number of (at least 2) rectangles, without overlaps, and in such a way that every rectangle has integer dimensions and a power of two as area. Show that two of the rectangles used must have the same dimensions. (Two rectangles are said to have the same dimensions if they have the same height and the same width, without rotating them.)

3. Find all positive integers k for which the equation

$$\operatorname{lcm}(m, n) - \gcd(m, n) = k(m - n)$$

does not have any solutions (m, n) in positive integers with $m \neq n$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xy - 1) + f(x)f(y) = 2xy - 1$$

for all $x, y \in \mathbb{R}$.

Solutions

1. The configuration is unique. As N is the midpoint of AC, we have $|NC| = |NA|$. Since $|NP| = |NC|$ by assumption, it follows that N is the centre of a circle passing through A, C, and P. Using Thales's theorem, we find $\angle APC = 90°$.

Since $|AH| = 3|BH|$ and M is the midpoint of AB, we have $|MH| = |BH|$. Moreover, because $\angle CHB = 90° = \angle CHM$, the triangles CHB and CHM are congruent, and therefore $|CM| = |CB|$. By assumption, $|CP| = |CB|$, so C is the centre of a circle through P, M, and B. As $\angle APC = 90°$, the line AP is tangent to this circle. By the tangent-chord theorem we now have $\angle APM = \angle PBM = \angle PBA$. $\square$

2. First note that a rectangle with integer dimensions has a power of two as area if and only if the dimensions are powers of two.

Consider a covering in which no two rectangles have the same dimensions. We first show that then there are no rectangles of width 1. Suppose to the contrary that such a rectangle occurs in the covering. Colour every square covered by a rectangle of width 1 with the colour blue. Let M be the number of blue squares. Then M is the sum of all rectangles of width 1, so it is the sum of (at least one) distinct powers of 2. Let 2^k be the largest one. As $2^k - 1 = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1$, there are fewer than 2^k blue squares not covered by this rectangle of width 1 and height 2^k, so there is at least one row which contains a blue square of this rectangle of height 2^k and no other blue square. But the remaining squares in this row, of which there are an odd number, must then be covered by rectangles of even width, which is a contradiction. So there are no rectangles of width 1. Analogously, there are no rectangles of height 1, either. Therefore all rectangles have an even width and an even height.

Now consider the smallest n for which a covering as in the problem exists in which no two rectangles have the same dimension. As all rectangles have an even width and an even height, we can divide all dimensions (of both the square and the rectangles) by 2 to obtain a square with smaller dimensions covered by rectangles with integer dimensions, each having a power of two as area. Since no two rectangles had the same dimensions, this contradicts the minimality of n. $\square$

3. Let $d = \gcd(m, n)$ and write $m = da$ and $n = db$. Then $\operatorname{lcm}(m, n) \cdot \gcd(m, n) = mn$, so we can rewrite the given equation as

$$\tfrac{da \cdot db}{d} - d = k(da - db),$$

or equivalently,

$$ab - 1 = k(a - b). \tag{1}$$

So from now on we consider the following equivalent problem: find all positive integers k for which (1) has no solutions (a, b) in positive integers with $a \neq b$ and $\gcd(a, b) = 1$. Note that if a pair (a, b) satisfies this equation, then it automatically follows that $\gcd(a, b) = 1$; suppose that $t \mid a$ and $t \mid b$, then we have $t \mid ab$ and $t \mid a - b$, so $t \mid 1$, and we deduce that a and b have no common divisor greater than 1.

First suppose that $k \geq 3$. We claim that $(a, b) = (k^2 - k - 1, k - 1)$ is a solution. Indeed, we have

$$ab - 1 = a(k - 1) - 1 = ka - a - 1 = ka - k^2 + k = k(a - k + 1) = k(a - b).$$

Moreover, as noted earlier, it follows that $\gcd(a, b) = 1$, so it remains to check that a and b are positive and distinct. As $k \geq 3$, we have $b = k - 1 \geq 2$, and $a = k^2 - k - 1 \geq 2k - k - 1 = k - 1 \geq 2$, so a and b are both positive. If $a = b$, then $k^2 - k - 1 = k - 1$, so $k^2 = 2k$, and therefore $k = 2$, which is a contradiction. Therefore (a, b) is as required. Hence (1) has a solution (a, b) in positive integers with $a \neq b$ and $\gcd(a, b) = 1$.

Now suppose that $k = 1$. We claim that $(a, b) = (2, 1)$ is a solution. Clearly, a and b are positive and distinct, and $\gcd(a, b) = 1$. Moreover, we have

$$ab - 1 = 2 - 1 = 1 = 1 \cdot (2 - 1) = k(a - b).$$

Hence (1) has a solution (a, b) in positive integers with $a \neq b$ and $\gcd(a, b) = 1$.

Finally, suppose that $k = 2$. Then the equation (1) becomes

$$ab - 1 = 2(a - b).$$

The right hand side is at most $2a - 2$ as b is a positive integer, so $ab - 1 \leq 2a - 2$, hence $ab < 2a$, and therefore $b < 2$. We deduce from this that $b = 1$. The equation then becomes $a - 1 = 2(a - 1)$, which implies $a - 1 = 0$. We therefore must have $a = 1$ and $b = 1$, so $a = b$. Therefore there are no solutions (a, b) for (1) in positive integers with $a \neq b$ and $\gcd(a, b) = 1$.

We conclude that the unique k for which the given equation has no solutions (m, n) in positive integers with $m \neq n$, is $k = 2$. $\square$

4. If f were constant, then the left hand side is constant, whereas the right hand side is not, this is a contradiction. Therefore f is not constant. Substitute $x = 0$. This gives $f(-1) + f(0)f(y) = -1$, so $f(0)f(y)$ is a constant function in y. As f is not constant, it follows that $f(0) = 0$, and therefore it also follows that $f(-1) = -1$. Substituting $x = y = 1$ gives: $f(0) + f(1)^2 = 1$, so we have either $f(1) = 1$ or $f(1) = -1$.

Now substituting $y = 1 + \frac{1}{x}$ with $x \neq 0$ gives $f(x+1-1) = f(x)f(1+\frac{1}{x}) = 2x + 2 - 1$, so we have

$$f(x)f\left(1 + \tfrac{1}{x}\right) = 2x + 1 - f(x) \qquad \text{for all } x \neq 0. \tag{2}$$

Substituting $y = \frac{1}{x}$ with $x \neq 0$ gives $f(1-1) + f(x)f(\frac{1}{x}) = 2 - 1$, so we have

$$f(x)f\left(\tfrac{1}{x}\right) = 1 \qquad \text{for all } x \neq 0. \tag{3}$$

Substituting $y = 1$, $x = z+1$ gives $f(z+1-1) + f(z+1)f(1) = 2z+2-1$, so we have

$$f(z) + f(z+1)f(1) = 2z + 1 \qquad \text{for all } z.$$

In this last equation, substitute $z = \frac{1}{x}$ with $x \neq 0$, and then multiply both sides with $f(x)$; we obtain

$$f(x)f\left(\tfrac{1}{x}\right) + f\left(\tfrac{1}{x} + 1\right)f(1)f(x) = \tfrac{2}{x}f(x) + f(x) \qquad \text{for all } x \neq 0.$$

Using (3) and (2), we can rewrite the first and second term respectively as follows:

$$1 + 2xf(1) + f(1) - f(x)f(1) = \tfrac{2}{x}f(x) + f(x) \qquad \text{for all } x \neq 0.$$

This we can rewrite as

$$f(x) \cdot \left(\tfrac{2}{x} + 1 + f(1)\right) = 1 + 2xf(1) + f(1) \qquad \text{for all } x \neq 0.$$

If the second factor of the left hand side is non-zero, then we can divide by this, so we obtain

$$f(x) = \frac{1 + 2xf(1) + f(1)}{\frac{2}{x} + 1 + f(1)} \qquad \text{if } x \neq 0 \text{ and } \tfrac{2}{x} + 1 + f(1) \neq 0.$$

Recall that we had two possible values for $f(1)$. First suppose that $f(1) = 1$. Then we have

$$f(x) = \frac{2 + 2x}{\frac{2}{x} + 2} = x \qquad \text{if } x \neq 0 \text{ and } \tfrac{2}{x} + 2 \neq 0.$$

Note that $\frac{2}{x} + 2 = 0$ only if $x = -1$. As we already know that $f(-1) = -1$ and $f(0) = 0$, it follows that $f(x) = x$ for all x. We check that this f satisfies the original equation: the left hand side then is $xy - 1 + xy = 2xy - 1$, so this f is indeed a solution.

Suppose on the other hand that $f(1) = -1$. Then we have

$$f(x) = \frac{-2x}{\frac{2}{x}} = -x^2 \qquad \text{if } x \neq 0 \text{ and } \tfrac{2}{x} \neq 0.$$

Note that $\frac{2}{x} = 0$ is impossible. Hence, as $f(0) = 0$, we deduce that $f(x) = -x^2$ for all x. We check that this f satisfies the original equation: the left hand side then is $-x^2y^2 - 1 + 2xy + x^2y^2 = 2xy - 1$, so this f is indeed a solution.

We conclude that there are two solutions, namely $f(x) = x$ for all x and $f(x) = -x^2$ for all x. $\square$

IMO Team Selection Test 2, June 2016

Problems

1. Prove that for all positive reals a, b, c we have:

$$a + \sqrt{ab} + \sqrt[3]{abc} \leq \tfrac{4}{3}(a + b + c).$$

2. Determine all pairs (a, b) of integers having the following property: there is an integer $d \geq 2$ such that $a^n + b^n + 1$ is divisible by d for all positive integers n.

3. Let $\triangle ABC$ be an isosceles triangle with $|AB| = |AC|$. Let D, E and F be points on line segments BC, CA and AB, respectively, such that $|BF| = |BE|$ and such that ED is the exterior angle bisector of $\angle BEC$. Prove that $|BD| = |EF|$ if and only if $|AF| = |EC|$.

4. Determine the number of sets $A = \{a_1, a_2, \ldots, a_{1000}\}$ of positive integers satisfying $a_1 < a_2 < \ldots < a_{1000} \leq 2014$, for which we have that the set

$$S = \{a_i + a_j \mid 1 \leq i, j \leq 1000 \text{ en } i + j \in A\}$$

is a subset of A.

Solutions

1. We can write $\sqrt[3]{abc}$ as $\sqrt[3]{\frac{a}{4} \cdot b \cdot 4c}$. Applying the inequality of the arithmetic and geometric mean on the positive reals $\frac{a}{4}$, b and $4c$ yields

$$\sqrt[3]{abc} = \sqrt[3]{\frac{a}{4} \cdot b \cdot 4c} \leq \frac{\frac{a}{4} + b + 4c}{3} = \frac{a}{12} + \frac{b}{3} + \frac{4c}{3}.$$

Next we apply AM-GM on $\frac{a}{2}$ and $2b$:

$$\sqrt{ab} = \sqrt{\frac{a}{2} \cdot 2b} \leq \frac{\frac{a}{2} + 2b}{2} = \frac{a}{4} + b.$$

We add up these two inequalities and we also add a to both sides to obtain:

$$a + \sqrt{ab} + \sqrt[3]{abc} \leq a + \frac{a}{4} + b + \frac{a}{12} + \frac{b}{3} + \frac{4c}{3} = \frac{4}{3}(a + b + c).$$

□

2. Consider a pair (a, b) that has the property, with the corresponding d. Let p be a prime divisor of d (which exists as $d \geq 2$). Because $d \mid a^n + b^n + 1$ for all n, we also have $p \mid a^n + b^n + 1$ for all n. Consider $n = p - 1$. Then $a^n \equiv 0 \mod p$ holds if $p \mid a$ and $a^n \equiv 1 \mod p$ holds if $p \nmid a$, because of the Fermat's little theorem. Similarly, this holds for b. Hence, $a^n + b^n + 1$ can attain the values 1, 2 and 3 modulo p. On the other hand, it must be congruent to 0 modulo p, hence $p = 2$ or $p = 3$. We consider the two cases.

Suppose that $p = 3$. Then we must have $3 \nmid a$, $3 \nmid b$. The case $n = 1$ moreover yields that $3 \mid a + b + 1$, hence $a + b \equiv 2 \mod 3$. These two requirements together are equivalent to $a \equiv b \equiv 1 \mod 3$. In this case, it is true that for all positive integers n we have $a^n + b^n + 1 \equiv 1 + 1 + 1 \equiv 0 \mod 3$, hence such a pair has the property, with $d = 3$.

Now suppose that $p = 2$. Then 2 must be a divisor of exactly one of a and b. In this case we have for all positive integers n that $a^n + b^n + 1 \equiv 0 + 1 + 1 \equiv 0 \mod 2$, hence each such pair has the property, with $d = 2$.

We conclude that these are the pairs that have the property: (a, b) with $a \equiv b \equiv 1 \mod 3$, (a, b) with $a \equiv 1$, $b \equiv 0 \mod 2$, and (a, b) with $a \equiv 0$, $b \equiv 1 \mod 2$. □

3. From the data and the angle bisector theorem it follows that

$$\frac{|BF|}{|BD|} = \frac{|BE|}{|BD|} = \frac{|CE|}{|CD|}.$$

As $\triangle ABC$ is isosceles, we have $\angle FBD = \angle ECD$, which yields together with the first equality that $\triangle BFD \sim \triangle CED$. This yields $\angle BFD = \angle CED = \angle BED$, hence $BDEF$ is a cyclic quadrilateral. It is well-known that a cyclic quadrilateral $BDEF$ is a trapezoid with $DE \parallel BF$ if and only if $|BD| = |EF|$. We have $|AF| = |EC|$ if and only if $|BF| = |AE|$ (as $|AB| = |AC|$), that is to say, if and only if $|EA| = |EB|$, which holds if and only if $\angle BAE = \angle ABE$. Because we have $2\angle BED = \angle BEC = \angle BAE + \angle ABE$ by the exterior angle theorem, $\angle BAE = \angle ABE$ is equivalent to $\angle BED = \angle ABE$ which is again equivalent to $DE \parallel BF$. To summarise, $|BD| = |EF|$ holds if and only if $DE \parallel BF$ which holds if and only if $|AF| = |EC|$. $\square$

4. We will prove that there are 2^{14} such sets. In particular, we will prove that the sets A that satisfy these conditions are of the form $B \cup C$, with C a subset of $\{2001, \dots, 2014\}$ and $B = \{1, 2, \dots, 1000 - |C|\}$. The sets of this form will be called "nice". As there are 2^{14} subsets of $\{2001, \dots, 2014\}$, there are 2^{14} nice sets.

First we will show that each nice set A satisfies the conditions. Suppose that $i, j \leq 1000$ with $i+j \in A$. Then we have $i+j \leq 2000$, hence $i+j \in B$. Therefore, there exists a k with $k \leq 1000 - |C|$ such that $i + j = a_k (= k)$. Because $a_k \leq 1000 - |C|$ holds, we also have $i, j \leq 1000 - |C|$, hence we have $a_i = i$ and $a_j = j$. That means that $a_i + a_j = i + j = a_k$ is an element of A. Hence each element of S is an element of A, from which we deduce that A satisfies the conditions.

We will now show that each set A satisfying the conditions is nice. First suppose that there exists an integer k with $1 \leq k \leq 1000$ satisfying $a_k \in \{1001, \dots, 2000\}$. Then we have $a_k = 1000 + i$ for a certain i with $i \leq 1000$, hence $a_{1000} + a_i$ is a an element of S and hence it must also be an element of A. However, we have $a_{1000} + a_i > a_{1000}$, which yields a contradiction. Hence such an a_k cannot occur. This means that A can be written as a disjoint union $B \cup C$, with $C \subseteq \{2001, \dots, 2014\}$ and $B \subseteq \{1, 2, \dots, 1000\}$. Let b be the number of elements of B. Then $b \geq 986$ holds, because C has at most 14 elements. In order to prove that A is nice, we must prove that $B = \{1, 2, \dots, b\}$. To prove this it is sufficient to prove that a_b, the maximum of B, equals b. Therefore, suppose the contrary, i.e. that $a_b > b$. For i with $i = a_b - b$ we then have $b + i = a_b \leq 1000$, hence $i \leq 1000 - b \leq 14 < b$. Therefore we have $a_i \leq 1000$ and hence $a_b + a_i \leq 2000$. Because $i + b = a_b \in A$, we have $a_b + a_i \in S \subset A$, but then $a_b + a_i$ is an element of B greater than a_b, which was the maximum. This is a contradiction. Hence, $a_b = b$, from which it follows that $B = \{1, 2, \dots, b\}$. Therefore, A is nice. $\square$

IMO Team Selection Test 3, June 2016

Problems

1. Let n be a positive integer. In a village, n boys and n girls are living. For the yearly ball, n dancing couples need to be formed, each of which consists of one boy and one girl. Every girl submits a list, which consists of the name of the boy with whom she wants to dance the most, together with zero or more names of other boys with whom she wants to dance. It turns out that n dancing couples can be formed in such a way that every girl is paired with a boy who is on her list.

Show that it is possible to form n dancing couples in such a way that every girl is paired with a boy who is on her list, and at least one girl is paired with the boy with whom she wants to dance the most.

2. For distinct real numbers $a_1, a_2, \ldots, a_n$, we calculate the $\frac{n(n-1)}{2}$ sums $a_i + a_j$ with $1 \leq i < j \leq n$, and sort them in ascending order. Find all integers $n \geq 3$ for which there exist $a_1, a_2, \ldots, a_n$ for which this sequence of $\frac{n(n-1)}{2}$ sums form an arithmetic progression (i.e. the difference between consecutive terms is constant).

3. Let k be a positive integer, and let $s(n)$ denote the sum of the digits of n. Show that among the positive integers with k digits, there are as many numbers n satisfying $s(n) < s(2n)$ as there are numbers n satisfying $s(n) > s(2n)$.

4. Let Γ_1 be a circle with centre A and Γ_2 be a circle with centre B, with A lying on Γ_2. On Γ_2 there is a (variable) point P not lying on AB. A line through P is a tangent of Γ_1 at S, and it intersects Γ_2 again in Q, with P and Q lying on the same side of AB. A different line through Q is tangent to Γ_1 at T. Moreover, let M be the foot of the perpendicular to AB through P. Let N be the intersection of AQ and MT.

Show that N lies on a line independent of the position of P on Γ_2.

Solutions

1. For each girl, call the boy with whom she wants to dance the most her *favourite*.

We solve the problem by induction on n. If $n = 1$, the only girl will form a couple with the only boy, who is therefore her favourite. So suppose that $k \geq 1$, and assume that the problem has been solved for $n = k$.

Consider the case $n = k + 1$. We distinguish two cases. First suppose that every boy occurs exactly once as a favourite. In this case we can just couple every girl to her favourite, and form n dancing couples that way.

In the remaining case, not every boy occurs exactly once as a favourite. Since there are n favourites and n boys, once of the boys, say X, is not the favourite of any girl (and someone else is the favourite of more than one girl). Choose a pairing as in the problem; this exists by assumption. Let Y be the girl coupled with boy X, and remove X and Y from the village. There are k boys and k girls left. Note that the pairing chosen still has the property that every girl is paired with a boy on her list. Moreover, every girls still has a favourite among the k remaining boys, as boy X is not the favourite of any girl. Therefore, by the induction hypothesis, we can form k dancing couples, in such a way that every girl is paired with a boy on her list, and at least one of the girls is paired with her favourite. Adding the couple $X - Y$ back in completes the induction. □

2. For $n = 3$ we consider $(a_1, a_2, a_3) = (1, 2, 3)$. The sums of pairs are in this case 3, 4, and 5, and these form an arithmetic progression. For $n = 4$ we consider $(a_1, a_2, a_3, a_4) = (1, 3, 4, 5)$. The sums of pairs are in this case 4, 5, 6, 7, 8, and 9, and these form an arithmetic progression.

Now suppose that $n \geq 5$, and suppose that $a_1, a_2, \ldots, a_n$ satisfies the condition. Without loss of generality, we assume that $a_1 < a_2 < \cdots < a_n$. Let d be the difference between two consecutive terms of the corresponding arithmetic progression. Note that the smallest sum is $a_1 + a_2$, and the second smallest is $a_1 + a_3$. As the difference between these sums is d, we have $a_3 - a_2 = d$. The largest sum is $a_n + a_{n-1}$ and the second largest is $a_n + a_{n-1}$, therefore we have $a_{n-1} - a_{n-2} = d$ as well. Hence

$$a_2 + a_{n-1} = (a_3 - d) + (a_{n-2} + d) = a_3 + a_{n-2}.$$

If $n \geq 6$, then the left hand side and the right hand side are sums of distinct pairs, but the difference between such sums must be at least d. This is a contradiction. Therefore there are no solutions for $n \geq 6$.

For $n = 5$, we have $a_3 - a_2 = d$ and $a_4 - a_3 = d$. Therefore the third smallest sum must be $a_1 + a_4$ (as this one is d larger than $a_1 + a_3$), and the third largest sum must be $a_5 + a_2$. Between these, we have $a_2 + a_3 < a_2 + a_4 < a_3 + a_4$, and the difference between these consecutive sums is d, and we also have $a_1 + a_5$. Therefore $a_1 + a_5$ is either the fourth smallest sum or the fourth largest sum.

Without loss of generality, assume that $a_1 + a_5$ is the fourth smallest sum. Then we have

$$\begin{aligned} a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < a_1 + a_5 < a_2 + a_3 \\ < a_2 + a_4 < a_3 + a_4 < a_2 + a_5 < a_3 + a_5 < a_4 + a_5. \end{aligned}$$

Then $(a_5 + a_2) - (a_3 + a_4) = d$, so $a_5 - a_4 = d + a_3 - a_2 = d + d$. On the other hand, $(a_1 + a_5) - (a_1 + a_4) = d$. This is a contradiction. Therefore there are no solutions for $n = 5$ either.

We conclude that there exist $a_1, a_2, \ldots, a_n$ satisfying the condition if and only if $n = 3$ or $n = 4$. □

3. We show that among the positive integers with *at most* k digits there are as many numbers n satisfying $s(n) < s(2n)$ as there are numbers satisfying $s(n) > s(2n)$. The required result then follows by combining this result for k and for $k - 1$.

We pair each number n with at most k digits to another number m with at most k digits. Let $m = 999 \cdots 999 - n$, where the first number consists of k nines, so that m has at most k digits. We show that $s(m) - s(2m) = s(2n) - s(n)$.

To calculate m, we subtract n from $999 \cdots 999$, in which every digit of n is of course at most 9. Therefore every digit of m is equal to 9 minus the corresponding digit of n. Here we consider both m and n as numbers having precisely k digits by adding zeroes to the left if necessary. Hence $s(m) + s(n) = s(999 \cdots 999) = 9k$.

Next, consider $2m$ and $2n$. We have $2m = 1999 \cdots 998 - 2n$. Consider $2m$ and $2n$ as numbers having exactly $k + 1$ digits, where the first digit of $2n$ is either a 0 or a 1. Subtracting that from $1999 \cdots 998$, we find that the first digit of $2m$ is either $1 - 1 = 0$ or $1 - 0 = 1$. The last digit of $2m$ is 8 minus the last digit of $2n$, which cannot be a 9 as $2n$ is even. All other digits of $2m$ are equal to 9 minus the corresponding digit of $2n$. Therefore, we have $s(2m) + s(2n) = s(1999 \cdots 998) = 1 + 9(k - 1) + 8 = 9k$. Hence $s(m) + s(n) = s(2m) + s(2n)$, so $s(m) - s(2m) = s(2n) - s(n)$.

Now we see that $s(m) > s(2m)$ if and only if $s(n) < s(2n)$. Moreover, no number is paired to itself as $999\cdots999$ is odd. Hence there are as many numbers with $s(n) < s(2n)$ as there are numbers with $s(n) > s(2n)$. $\square$

4. Point P lies outside Γ_1, since otherwise there is no tangent PS to Γ_1. Since P and Q lie on the same side of AB, we see that S lies on the part of Γ_1 on that same side of AB, and that S lies outside Γ_2. (In the extremal case in which P lies on AB, we see that S an intersection point of Γ_1 and Γ_2 by Thales's theorem.) Consider the configuration in which Q lies between P and S; then Q lies on the short arc AP. The other configuration is treated analogously. (Note that by assumption $P \neq Q$.)

We show that N lies on the radical line of Γ_1 and Γ_2. We have $\angle ASP = 90° = \angle AMP$, so $ASPM$ is a cyclic quadrilateral by the Thales's theorem. Hence we have

$$\begin{aligned}\angle PSM = \angle PAM = \angle PAB &= 90° - \tfrac{1}{2}\angle ABP \\ &= 90° - (180° - \angle AQP) = 90° - \angle AQS,\end{aligned}$$

where in the second to last step, we applied the inscribed angle theorem. Moreover, using the sum of angles in $\triangle AQS$, we find that $90° - \angle AQS = \angle QAS$. As $ASQT$ is a cyclic quadrilateral with $|QT| = |QS|$ (since QS and QT are both tangent to Γ_1), we have $\angle QAS = \angle QTS = \angle QST$. To summarise, we have $\angle PSM = 90° - \angle AQS = \angle QAS = \angle QST = \angle PST$. Hence S, T, and M are collinear.

From this, it follows that N is the intersection of ST and AQ. In the cyclic quadrilateral $ASQT$ we find, using the power of a point theorem, that $NT \cdot NS = NA \cdot NQ$. Note that the left hand side of N is the power of N with respect to Γ_1 and that the right hand side is the power of N with respect to Γ_2. Therefore N lies on the radical line of Γ_1 and Γ_2. $\square$

Junior Mathematical Olympiad, October 2015

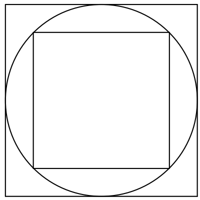
Problems

Part 1

1. A booklet is made by forming a stack of 11 sheets of paper and then folding the stack in half. The pages of the booklet are numbered, like in a book, from 1 to 44, where the front cover gets the number 1 and the back cover gets number 44. Now the booklet is opened up and from the stack of 11 sheets we take the one in the middle.
Adding up the four numbers on this sheet, what outcome do we get?

A) 82 B) 84 C) 86 D) 88 E) 90

2. We draw a circle through the four vertices of a square of area 1. Then, we draw a square around this circle in such a way that that all the sides are tangent to the circle.
What is the area of this square?



A) $\frac{10}{7}$ B) $\frac{3}{2}$ C) $\frac{5}{3}$ D) 2 E) $\frac{100}{49}$

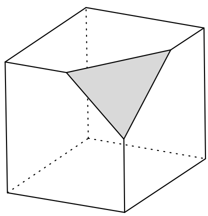
3. Quintijn has three equally big and equally filled bottles of wine. Bottles 1 and 3 contain the same kind of white wine, while bottle 2 contains red wine. Quintijn now pours a small amount of wine from bottle 1 into bottle 2. Next, after mixing the content of bottle 2 really well, he pours the same amount from bottle 2 into bottle 3. In the same way, he pours the same amount of wine from bottle 3 into bottle 1. Now all bottles contain the same amount of wine as they did at the start. However, the content of each bottle is polluted with wine of the other type.
Which bottle is polluted most?

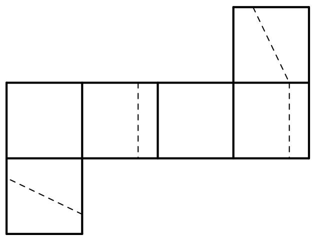
A) bottle 1 B) bottle 2 C) bottle 3
D) all three equally E) you cannot determine this

4. Aad, Bep, Cor, Dirk, Eva, and Fenna are sitting in this order in a circle around the campfire. Aad has a torch. He gives it to Bep who is sitting one place to his right. She gives the torch to Dirk, who is sitting two places to her right. He gives the torch to Aad, who is sitting three places to his right, et cetera. It happens that someone must give the torch to the person sitting, for instance, six or twelve positions to the right. Then this person gives the torch to him- or herself.
When Dirk is given the torch for the hundredth time, who does he pass the torch to?

A) Aad B) Bep C) Cor D) Dirk E) Eva

5. When a cube is cut with a plane, a cross section is created. This figure is formed by the lines where the plane cuts the facets of the cube. In the left figure, you can see an example in which the cross section is a triangle.





In the right figure, a flattened cube is drawn with the cut lines drawn on the facets.
What is the cross section of the cube corresponding to this figure?

A) a triangle B) a square C) a rectangle, but not a square
D) a hexagon E) a parallelogram, but not a rectangle

6. On a machine there are three buttons. The first button can be used to add 20 marbles to a tray in the machine. The second button can be used to increase the number of marbles in the tray by 20%, after which 15 additional marbles are added. The third button can be used to increase the number of marbles in the tray by 50%. If pressing a button would cause the number of marbles to be non-integral, the pressing of the button is not allowed. In the beginning the tray is empty. After some button presses, there are 91 marbles in the tray.
How often has the first button been pressed?

A) 0 B) 1 C) 2 D) 3 E) 4

7. Ria, Sophie, and Tine are sitting around a round table in clockwise order and are playing a game with chips. Ria starts with 3 chips, Sophie with 4 chips, and Tine with 5 chips. In each round they simultaneously give chips to one of their neighbours. Each player can choose to give 2 chips to her right neighbour or 1 chip to her left neighbour. If someone has no more chips, the game ends.
Is it possible for the players to have the same number of chips after some number of rounds, and if this is the case, how many rounds have to be played at least to accomplish this?

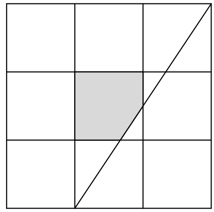
A) no, it is impossible. B) yes, 3 rounds. C) yes, 6 rounds.
D) yes, 7 rounds. E) yes, 8 rounds.

8. Six people are sitting around a round table. Each of them is either a knight or a knave. Knights always speak the truth, while knaves always lie. Each of them has a card containing a number. All numbers are different and everyone knows the numbers of their two neighbours. When asked: "Is your number greater than the numbers of both your neighbours?", everyone answers with "Yes". When asked: "Is your number smaller than the numbers of both your neighbours?", at least one person answers "Yes" and at least one answers "No".
What are the possible numbers of people answering "Yes" to this second question?

A) 1 or 2 B) 1 or 3 C) 2 or 3 D) 2 or 4 E) 2, 3, or 4

Part 2

1. In the figure on the right there is a square with side length 3. The square is divided into nine equal squares. Then, another line is drawn that creates a pentagon inside the middle square (coloured grey).
What is the area of this pentagon?



2. A palindromic number is a number that is the same when read from left to right as when read from right to left. A number does not start with the digit 0. To a six-digit palindromic number the palindromic number 21312 is added. The result is a seven-digit palindromic number.
What is this resulting number?

3. Using exactly six zeros and six ones we create two or more numbers which we then multiply. For instance, we could get $10 \times 10 \times 10 \times 10 \times 10 \times 10 =$ 1,000,000, or $10{,}011 \times 100 \times 1{,}100 = 1{,}101{,}210{,}000$.
What is the largest possible result we could get in this way?

4. Pieter is staying at a hotel. The hotel has a ground floor (numbered 0) and seven additional floors numbered 1 to 7. Pieter wants to make a trip by elevator. He starts on one of the floors 1 to 7 and ends at the ground floor. In between, he travels from floor to floor, never stopping at a previously visited floor and never stopping at the ground floor (except for the last stop).

The distance between any two consecutive floors is 3 metres. If, for example, Pieter starts at floor 3, then goes to floor 6, then to floor 4 and finally to the ground floor, he travels a total of $(3+2+4) \times 3 = 27$ metres.
What is the maximum length of his trip in metres?

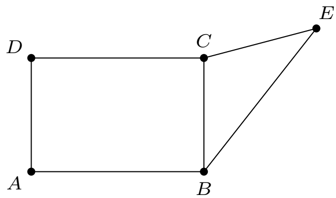
5. How many 3-digit numbers (the first digit cannot be 0) have the property that adding all the digits gives a strictly greater result than multiplying all the digits?

6. The mayor of a town wants to build a network of express trams. She wants it to meet the following conditions:

- There are at least two distinct tram lines.
- Each tram line serves exactly three stops (also counting the start and terminus).
- For each two tram stops in the town there is exactly one tram line that serves both stops.

What is the minimum number of stops that the mayor's tram network can have?

7. A rectangle $ABCD$ and a point E are given. Line segments BE and BA have the same length. Line segments CE and CB also have the same length. Moreover, the area of rectangle $ABCD$ is four times as large as the area of triangle BCE. Side AD has length 10. What is the length of diagonal AC?



8. We consider ways to fill a 5×5-board by writing a 1 or a 3 in each square. Such a filling is called *balanced* if the following holds:

- If you take an arbitrary 3×3-square of the board and multiply all the numbers that it contains, and after that you do the same for an arbitrary 4×4-square, then the second result is always three times as large as the first result.

In the figure on the right, you see a filling that is *not* balanced. For example, when multiplying the numbers in the indicated 4×4-square, the result is *nine times* as large as the result obtained by multiplying the numbers in the indicated 3×3-square.

1	1	3	3	1
3	1	1	1	3
1	3	1	3	1
3	1	1	3	1
1	1	3	1	1

Give (on the answer form) a balanced filling of the board containing a maximum number of 3-s.

Solutions

Part 1

1. E) 90

2. D) 2

3. B) bottle 2

4. B) Bep

5. C) a rectangle, but not a square

6. D) 3

7. A) no, this is impossible.

8. D) 2 or 4

Part 2

1. $\frac{11}{12}$

2. 1008001

3. 12,321,000,000

4. 84

5. 199

6. 7

7. 20

8. There are two possibilities:

1	1	1	1	1
1	1	1	1	1
3	3	3	3	3
1	1	1	1	1
1	1	1	1	1

1	1	3	1	1
1	1	3	1	1
1	1	3	1	1
1	1	3	1	1
1	1	3	1	1

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