Second round
Dutch Mathematical Olympiad

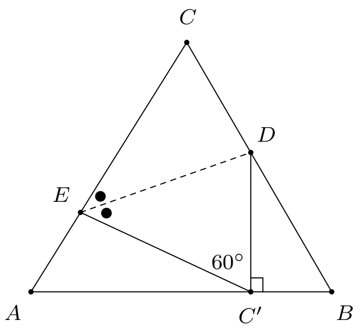
Friday 14 March 2014

Solutions

B-problems

B1. 169 Brenda can make 25 differently filled pouches without blue marbles as she can put 1 to 25 red marbles in a pouch. There are 23 differently filled pouches possible containing 1 blue marble because 2 to 24 red marbles may be added. Using 2 blue marbles there are 21 possibilities, namely by adding 3 to 23 red marbles. In total, there are $25 + 23 + \ldots + 1 = 169$ differently filled pouches that Brenda can make.

B2. 45° First, we notice that triangles CDE and $C'DE$ are each other's mirror image and hence have equally sized angles. In particular, we have $\angle DC'E = \angle DCE = 60°$. Furthermore, we have $\angle CED = \angle DEC'$, see the figure.



From $\angle AC'B = 180°$ it follows that $\angle AC'E = 180° - 90° - 60° = 30°$. The sum of the angles in triangle $AC'E$ is 180° and hence we find $\angle AEC' = 180° - 60° - 30° = 90°$. From $\angle AEC = 180°$ it follows that $\angle CEC' = 180° - 90° = 90°$.
We conclude that $\angle DEC' = \frac{1}{2}\angle CEC' = 45°$.

B3. 13 If $8n + 1$ is the square of an integer, then this integer must be odd. Conversely, the square of an odd integer is always a multiple of 8 plus 1. Indeed, suppose that k is an odd integer, then we may write $k = 2\ell + 1$ for an integer ℓ. We see that

$$k^2 = (2\ell + 1)^2 = 4\ell^2 + 4\ell + 1 = 4\ell(\ell + 1) + 1.$$

Because either ℓ or $\ell + 1$ is even, we deduce that $4\ell(\ell + 1)$ is divisible by 8. Hence, k^2 is a multiple of 8 plus 1.

As a result, we only need to determine the number of odd squares x for which $8 \cdot 1 + 1 \leqslant x \leqslant 8 \cdot 100 + 1$. These are the squares $3^2 = 9$, $5^2 = 25$ up to and including $27^2 = 729$, because $29^2 = 841$ is greater than 801. Thus, the number of squares of the desired form is 13.

B4. 5 The number that Evan came up with is denoted by c_{10}, the number of his right neighbour is denoted by c_9, continuing in this way until the left neighbour of Evan, whose number is denoted by c_1. From the data we deduce that

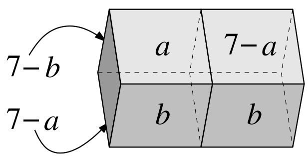
$$\begin{aligned} c_{10} + c_8 &= 2 \cdot 9 = 18, \\ c_8 + c_6 &= 2 \cdot 7 = 14, \\ c_6 + c_4 &= 2 \cdot 5 = 10, \\ c_4 + c_2 &= 2 \cdot 3 = 6, \\ c_2 + c_{10} &= 2 \cdot 1 = 2. \end{aligned}$$

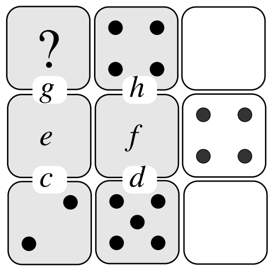
Adding up these equations yields $2(c_2 + c_4 + c_6 + c_8 + c_{10}) = 50$, hence $c_2 + c_4 + c_6 + c_8 + c_{10} = 25$. Finally, we find $c_{10} = (c_2 + c_4 + c_6 + c_8 + c_{10}) - (c_2 + c_4) - (c_6 + c_8) = 25 - 6 - 14 = 5$.

B5. 3 The numbers of dots on opposite faces of a die will be called *complementary.* Together, they always add to 7. Consider a pair of dice that touch in faces with equal numbers of dots. We still allow them to rotate with respect to each other. When rotating the dice, their faces show the same numbers of dots, but in reverse cyclic order.

Consider the situation where the numbers of dots on the top faces of the dice are complementary, say a and $7 - a$. This is depicted in the figure on the left. On the four faces around the gluing axis, the left hand die will have a, b, $7 - a$, and $7 - b$ dots in this order, for some b. Hence, the right hand die has these numers in reverse cyclic order: $7 - a$, b, a, and $7 - b$ dots on the corresponding faces. It follows that the dice have the same number of dots on the front face, namely b, and the same number of dots on the back face, namely $7 - b$.

Conversely, if two glued dice have the same number of dots on the front faces (or back faces), then the numbers of dots on the top faces must be complementary.





We will now apply this to the six dice in the first two columns of the 3×3 array, see the figure on the right. The two dice in the third row show complementary numbers of dots on their top faces, namely 2 and 5. Therefore, the numbers of dots c and d, on the faces where the dice are glued to dice in the second row, must be equal. This in turn implies that the numbers of dots e and f on the top faces must be complementary. Therefore, the numbers g and h are equal. Finally, the top faces of the dice in the first row must show complementary numbers of dots. Hence, there must be 3 dots on the place of the question mark.

C-problems

C1. (a) First, we notice that $a + d$ is the area of triangle AEF and $p + q$ is the area of triangle BEF. The base AE of triangle AEF has the same length as the base BE of triangle BEF. Because the two triangles have the same height, they must also have equal areas.

(b) Here we use that triangles DEF and CEF have bases of the same length ($|DF| = |CF|$), and equal corresponding heights. Hence, they have equal areas. This implies that $c + d = q + r$. Subtracting the equation of part (a) yields $c - a = r - p$, or $a + r = c + p$, as required.

(c) The heights of triangles AED, BEC and ABF with respect to the bases AE, BE, and AB are denoted by x, y, and z. Because F is the midpoint of CD, the height z is the average of the heights x and y, in formulas: $\frac{x+y}{2} = z$. The area of triangle AED is $\frac{1}{2} \cdot x \cdot |AE|$, the area of triangle BEC is $\frac{1}{2} \cdot y \cdot |BE|$, and the area of triangle ABF is $\frac{1}{2} \cdot z \cdot |AB|$. Because E is the midpoint of AB we have $|AE| = |BE|$ and $|AB| = 2 \cdot |AE|$. The sum of the areas of triangles AED and BEC is thus equal to $\frac{1}{2} \cdot (x + y) \cdot |AE| = z \cdot |AE|$, while the area of triangle ABF is equal to $\frac{1}{2} \cdot z \cdot 2|AE| = z \cdot |AE|$. Hence, these areas are equal and we find $a + b + p + s = a + d + p + q$. By subtracting $a + p$ on both sides of the equation, we find $b + s = d + q$, as required.

C2. (a) Consider the ten digit number 1001001001. If we multiply it with 111, we get the number 111111111111 which consists of ones only. Hence, the number 1001001001 is a jackpot number.

(b) Let k be a number of at least two digits, all of which are equal, say equal to a. Remark that $a \neq 0$. We have to prove that the digits of the number

$$11k \;=\; k + 10k \;=\; a\cdots a \;+\; a\cdots a0$$

are not all equal.

The last digit of $11k$ is a. We will show that the second last digit of $11k$ is unequal to a. There are two cases. If $a \leqslant 4$, then the second last digit of $11k$ is equal to $a + a$. This is unequal to a as $a \neq 0$. If $a \geqslant 5$, then the second last digit of $11k$ is equal to $a + a - 10$. This is unequal to a as $a \neq 10$. We conclude that 11 is not a jackpot number.

(c) That 143 is a jackpot number follows directly from the fact that $143 \cdot 777 = 111111$.