Second round
Dutch Mathematical Olympiad

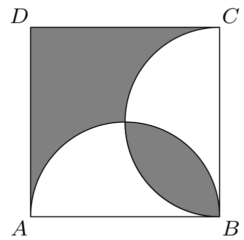
Friday 15 March 2013

- Time available: 2.5 hours.
- The competition consists of five B-problems and two C-problems.
- Formula sheets and calculators are not allowed. You can only use a pen, compass, ruler or set square and of course your mental skills.
- Good luck!

B-problems

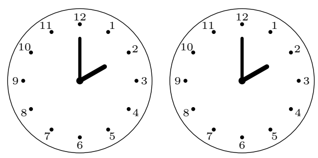
The answer to each B-problem is a number. A correct answer is awarded 4 points, for a wrong answer no points are given. Please work very accurately: a minor error in a calculation may result in a wrong answer. NOTE: all answers should be given in exact form, like $\frac{11}{81}$ or 5^8 or $\frac{1}{4}(\sqrt{5}+\pi)$.

B1. A number of students took a test for which the maximum possible score was 100 points. Everyone had a score of at least 60 points. Exactly five students scored the maximum of 100 points. The average score among the students was 76 points.
What is the minimum number of students that could have taken the test?



B2. In the figure, a square $ABCD$ of side length 4 is given. Inside the square, two semicircles with diameters AB and BC are drawn.
Determine the combined area of the two grey shapes.

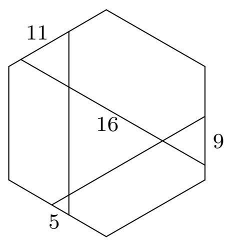
B3. Consider two clocks, like the ones in the figure on the right, whose hands move at a constant speed. Both clocks are defective; the hands of the first clock turn at a pace that is 1% faster than it should be, while the hands of the second clock turn at a pace that is 5% too fast. At a certain moment, both clocks show a time of exactly 2 o'clock. Some time passes until both clocks again show exactly the same time.
At that moment, what time do the clocks show?



B4. The number square on the right is filled with positive numbers. The *product* of the numbers in each row, in each column, and in each of the two diagonals is always the same.
What number is H?

$\frac{1}{2}$	32	A	B
C	2	8	2
4	1	D	E
F	G	H	16

B5. A regular hexagon is divided into seven parts by lines parallel to its sides, see the figure. Four of those pieces are equilateral triangles, whose side lengths are indicated in the figure.
What is the side length of the regular hexagon?



PLEASE CONTINUE ON THE OTHER SIDE

C-problems

For the C-problems not only the answer is important; you also have to write down a clear reasoning. Use separate sheets of paper for each C-problem. A correct and well-explained answer is awarded 10 points.
Partial solutions may also be worth some points. Therefore, write neatly and hand in your drafts (for each problem separately).

C1. We say a positive n-digit number ($n \geqslant 3$ and $n \leqslant 9$) is *above average* if it has the following two properties:

- the number contains each digit from 1 to n exactly once;
- for each digit, except the first two, the following holds: twice the digit is at least the sum of the two preceding digits.

For example, 31254 is above average because it consists of the digits 1 to 5 (each exactly once) and also

$$2 \cdot 2 \geqslant 3 + 1, \qquad 2 \cdot 5 \geqslant 1 + 2, \qquad \text{and} \qquad 2 \cdot 4 \geqslant 2 + 5.$$

(a) Give a 4-digit number that is above average and has '4' as its first digit.

(b) Show that no 4-digit number that is above average has '4' as its second digit.

(c) For 7-digit numbers that are above average, determine all possible positions of the digit '7'.

C2. We will call a triple (x, y, z) *good* if x, y, and z are positive integers such that $y \geqslant 2$ and the equation $x^2 - 3y^2 = z^2 - 3$ holds.
An example of a good triple is $(19, 6, 16)$, because $6 \geqslant 2$ and $19^2 - 3 \cdot 6^2 = 16^2 - 3$.

(a) Show that for every *odd* number $x \geqslant 5$ there are at least two good triples (x, y, z).

(b) Find a good triple (x, y, z) with x being *even*.