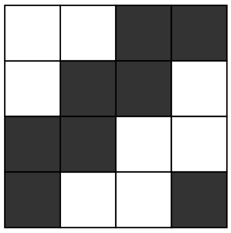
Final round
Dutch Mathematical Olympiad

Friday 13 September 2013

Solutions

1. We will prove that $n = 4$ is the largest possible n for which an $n \times n$-table can be coloured according to the rules. The following figure shows a valid colouring for $n = 4$.



Now we prove that there is no colouring of the squares in a 5×5-table satisfying the requirements. Suppose, for contradiction, that such a colouring exists. Of the squares in each row either the majority is black, or the majority is white. We may suppose that there are at least three rows for which the majority of the squares is black (the case where there are at least three rows for which the majority of the squares is white is treated in an analogous way). We now consider the squares in these three rows. Of these 15 squares at least 9 are black.

If there is a column in which each of the three rows has a black square, then each other column can contain at most one black square in these three rows. The total number of black squares in the three rows will therefore be no more than $3 + 1 + 1 + 1 + 1 = 7$, contradicting the fact that the number should be at least 9.

Hence, in each column at most two of the three rows have a black square. We consider the number of columns with two black squares in the three rows. If there are more then three, then there are two columns in which the same two rows have a black square, which is impossible. It follows that the number of black squares in the three rows is no more than $2+2+2+1+1 = 8$, again contradicting the fact that this number should be at least 9.

Hence, it is impossible to colour a 5×5-table according to the rules. It is clear that it also will be impossible to colour an $n \times n$-table according to the rules if $n > 5$. $\square$

2. The first equation yields $z = x + y + 1$. Substitution into the second equation gives $x^2 - y^2 + (x + y + 1)^2 = 1$. Expanding gives $2x^2 + 2xy + 2x + 2y = 0$, or $2(x + y)(x + 1) = 0$. We deduce that $x + y = 0$ or $x + 1 = 0$. We consider the two cases.

- If $x + y = 0$, then $y = -x$ holds. The first equation becomes $z = 1$. Substitution into the third equation yields $-x^3 + (-x)^3 + 1^3 = -1$, or $x^3 = 1$. We deduce that $(x, y, z) = (1, -1, 1)$.
- If $x + 1 = 0$, then $x = -1$ holds. The first equation becomes $z = y$. Substitution into the third equation yields $-(-1)^3 + y^3 + y^3 = -1$, or $y^3 = -1$. Hence, we have $(x, y, z) = (-1, -1, -1)$.

In total we have found two possible solutions. Substitution into the original equations shows that these are indeed both solutions to the given system. $\square$

3. First, we prove that some triangles in the figure are isosceles (the top angle coincides with the middle letter).

(1) Triangle ADC is isosceles, because $\angle DAC = \angle ACB = \angle ACD$. The first equality holds because AD and BC are parallel and the second equality follows from the fact that AC is the interior angle bisector of angle BCD.

(2) Triangle DAO is isosceles, because $|AD| = |CD| = |AO|$. The first equality follows from (1) and the second equality is given in the problem statement.

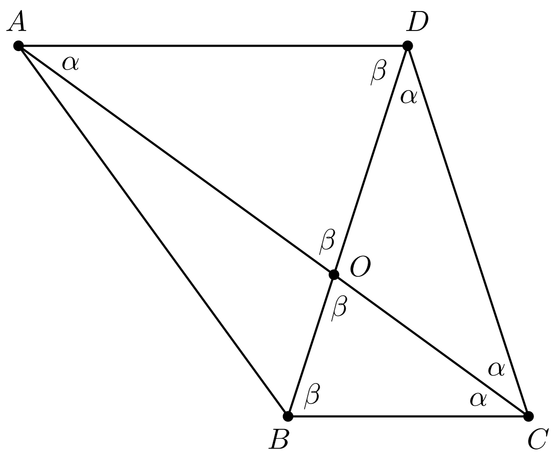
(3) Triangle BCO is isosceles, because it is similar to triangle DAO (hourglass shape).

(4) Triangle COD is isosceles, because $|DO| = |BC| = |CO|$. Here the first equality is given in the problem statement and the second one follows from (3).

(5) Triangle BDC is isosceles, because it is similar to triangle BCO as two pairs of corresponding angles are equal: $\angle DBC = \angle OBC$ and $\angle BDC = \angle DCO = \angle OCB$. Here the last equality follows from (4).

(6) Triangle ADB is isosceles, because $|AD| = |CD| = |BD|$ because of (1) and (5).

Denote $\angle ACB = \alpha$ and $\angle CBD = \beta$ (in degrees). From (5) it follows that $2\alpha = \beta$. From (3) it follows that $180° = 2\beta + \alpha = 5\alpha$, hence $\alpha = \frac{180°}{5} = 36°$ and $\beta = 72°$. In the isosceles triangle ADB the top angle is equal to $\angle ADB = \beta$, hence its equal base angles are $\frac{180°-72°}{2} = 54°$. The requested angle therefore equals $\angle ABC = \angle ABD + \angle DBC = 54° + 72° = 126°$.



□

4. a) Because $P(n) = 15n$ is the product of the positive divisors of n, the prime divisors 3 and 5 of $P(n)$ must also be divisors of n. It follows that n is a multiple of 15. If $n > 15$, then 3, 5, 15 and n are distinct divisors of n, yielding $P(n) \geqslant 3 \cdot 5 \cdot 15 \cdot n = 225n$. This contradicts the fact that $P(n) = 15n$. The only remaining possibility is $n = 15$. This is indeed a solution, because $P(15) = 1 \cdot 3 \cdot 5 \cdot 15 = 15 \cdot 15$.

b) Suppose that $P(n) = 15n^2$ holds. Again, we find that n is a multiple of 15. It is clear that $n = 15$ is not a sulution, hence $n \geqslant 30$. We observe that $\frac{n}{5} > 5$. It follows that $1 < 3 < 5 < \frac{n}{5} < \frac{n}{3} < n$ are six distinct divisors of n. Thus $P(n) \geqslant 1 \cdot 3 \cdot 5 \cdot \frac{n}{5} \cdot \frac{n}{3} \cdot n = n^3$. Because $n > 15$ holds, we have $P(n) \geqslant n \cdot n^2 > 15n^2$, which contradicts the assumption of this problem. We conclude that no n exists for which $P(n) = 15n^2$. $\square$

5. *To illustrate the idea, we first calculate the number of fives in the result s of the sum $1 + 10 + 19 + \cdots + 100000$. First notice that each term in the sum is a multiple of nine plus 1:*

$$s = 1 + (1 + 9) + (1 + 2 \cdot 9) + \cdots + (1 + 11111 \cdot 9).$$

The number of terms in the sum is $11111 + 1 = 11112$ and the average value of a term is $\frac{1+100000}{2}$. It follows that $s = 11112 \cdot \frac{100001}{2} = \frac{11112}{2} \cdot 100001$.
Because $\frac{11112}{2} = \frac{5 \cdot 11112}{5 \cdot 2} = \frac{55560}{10} = 5556$, we find that $s = 5556 + 555600000 = 555605556$. Hence, the number of fives is equal to 6 in this case.

Now we will solve the actual problem. Remark that $10^{2013} = 1 + 9 \cdot 11\ldots1$ (2013 ones). For simplicity let $n = 11\ldots1$ be the number consisting of 2013 ones. We see that the sum

$$S = 1 + (1 + 9) + (1 + 2 \cdot 9) + \cdots + (1 + n \cdot 9)$$

has exactly $n + 1$ terms, with an average value of $\frac{1+10^{1023}}{2}$. Hence $S = \frac{n+1}{2} \cdot (1 + 10^{2013})$.
Calculating the fraction $\frac{n+1}{2}$ gives:

$$\frac{n+1}{2} = \frac{5n+5}{10} = \frac{555\ldots560}{10} = 555\ldots56,$$

a number with 2011 fives followed by a 6. Because the last 2013 digits of the number $10^{2013} \cdot \frac{n+1}{2}$ are all zeroes, there is no 'overlap' between the non-zero digits of $\frac{n+1}{2}$ and $10^{2013} \cdot \frac{n+1}{2}$. We deduce that

$$\begin{aligned} S &= \tfrac{n+1}{2} \cdot (1 + 10^{2013}) \\ &= \tfrac{n+1}{2} + 10^{2013} \cdot \tfrac{n+1}{2} \\ &= 55\ldots56055\ldots56. \end{aligned}$$

Hence S is a number that contains exactly $2011 + 2011 = 4022$ fives. $\square$