Final round
Dutch Mathematical Olympiad

Friday 14 September 2012

Solutions

1. In order to show that the product

$$n = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12, it suffices to show that it is divisible by 3 and by 4. We consider the remainder upon division by 3 for the numbers a, b, c, and d. Two of these numbers must have the same remainder since there are four numbers and only three possible remainders. The difference of these two numbers must then be divisible by 3, which implies that also n is divisible by 3.

If among the numbers a, b, c, and d there are at least three numbers of the same parity (all three *even* or all three *odd*), this implies that the three pairwise differences between these numbers are even. Hence n is divisible by $2\times2\times2 = 8$.

Otherwise, two of the numbers are *even* and the other two are *odd*. Both pairwise differences are *even*, which shows that n is divisible by $2\times2 = 4$. □

2. (a) Yes, it can be done. A possible filling of the 5×5-board containing exactly two blue cells in each row can be seen in the figure below.

1	2	3	4	5
5	4	3	2	1
1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5

(b) Consider a 10×10-board that is filled in accordance to the given rules. Every column now contains the numbers 1 to 10. In column 1, there are 9 blue cells (those containing numbers '2' to '10'), in column 2 there are 8 blue cells (containing '3' to '10'), in column 3 there are 7 blue cells (containing '4' to '10'), etc. In total, there are $9+8+7+\cdots+0 = 45$ blue cells. Because 45 is not divisible by 10, it is impossible for all rows to have the same number of blue cells.

□

3. All terms containing a factor p are brought to the left-hand side of the equation. In this way we obtain $p^3 + mp - p = m^2 - 2m + 1$, or

$$p(p^2 + m - 1) = (m-1)^2.$$

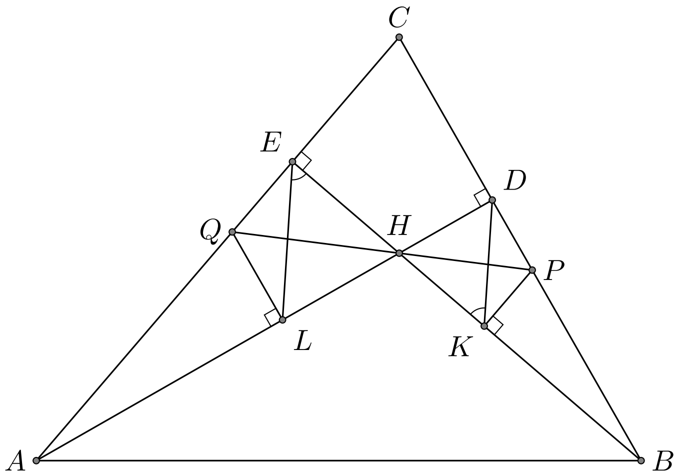
Note that $m-1$ is nonnegative by assumption. Observe that p is a divisor of $(m-1)^2$. Since p is a prime number, p must also divide $m-1$. We may write $m-1 = kp$, with k a nonnegative integer. Substituting this into the previous equation gives: $p(p^2+kp) = k^2p^2$. Dividing by p^2 on both sides, we find $p + k = k^2$, or

$$p = k(k-1).$$

As p is prime, one of the factors k and $k-1$ must equal 1 (the case $k-1=-1$ is excluded). The case $k=1$ does not lead to a solution because then $k-1=0$. Hence, we must have $k-1=1$, which gives the only candidate for a solution $k=2$, $p=2$ en $m=5$, and hence $(p,m) = (2,5)$. It is clear that this is indeed a solution. □

4. We will use similarity of certain triangles. Observe that $\angle DHP = \angle LHQ$ (opposite angles), and that $\angle PDH = 90^\circ = \angle QLH$. It follows that triangles DHP and LHQ are similar (AA). This implies that $\frac{|DH|}{|LH|} = \frac{|HP|}{|HQ|}$. In the same way, we can see that triangles KHP and EHQ are similar, which implies that $\frac{|KH|}{|EH|} = \frac{|HP|}{|HQ|}$.

Combining these two inequalities shows that $\frac{|DH|}{|LH|} = \frac{|KH|}{|EH|}$. Because $\angle DHK = \angle LHE$ (opposite angles), this implies that triangles DHK and LHE are similar (SAS). Similarity of these triangles implies that $\angle HKD = \angle HEL$, from which we may conclude (corresponding angles) that DK and EL are parallel.



□

5. A sequence meeting the demand is called 'good'. We will determine the number of good sequences in an indirect way, by first counting something else.

First consider the number of ways in which we can colour the numbers 1 to 12, each number being coloured either red or blue. As there are two options (red or blue) for each of the 12 numbers independently, there are 2^{12} possible colourings. We call a colouring 'good' if there is at least one number of each colour, and the largest red number is larger than the smallest blue number. There are exactly 13 colourings that are *not* good: the colouring having only blue numbers, and the twelve colourings in which the red numbers are precisely the numbers 1 to k, for some $k = 1, 2, \ldots, 12$ (in the case $k = 12$, there are no blue numbers.) In total, there are $2^{12} - 13 = 4083$ good colourings.

We will now show that the number of good sequences is equal to the number of good colourings, by perfectly matching colourings and sequences. Take a good sequence and suppose that a is the number smaller than its immediate predecessor in the sequence. To this sequence we associate the following colouring: the numbers preceding a in the sequence are coloured red, and the other numbers are coloured blue. This will be a good colouring.

As an example, consider the good sequence $1, 3, 4, 5, 8, 9, 2, 6, 7, 10, 11, 12$. Hence, $a = 2$. The numbers 1, 3, 4, 5, 8, and 9 will be coloured red, the numbers 2, 6, 7, 10, 11, and 12 will be blue. This is a good colouring, because $9 > 2$.

In this way, each good colouring is obtained precisely once from a good sequence. Indeed, for a given good colouring, we can find the unique corresponding good sequence as follows: write down the red numbers in increasing order, followed by the blue numbers in increasing order.

We conclude that there are 4083 sequences meeting the demand. □