
# $50^{th}$ Dutch Mathematical Olympiad 2011

NEDERLANDSE
WISKUNDE
OLYMPIADE

# Contents



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# Introduction

In 2011 the Dutch Mathematical Olympiad celebrated its 50th anniversary. The most special fact in this anniversary year was that we hosted the International Mathematical Olympiad (IMO) for the first time. From 16 until 24 July, 564 contestants from 101 countries came to Amsterdam to test their mathematical talents and to enjoy the social events. More than 300 guides, invigilators, coordinators and many other organisers worked together to make the event a big success.

In the meantime the entire selection process for IMO 2012 started with the first round on 4 February 2011, held at the participating schools. The paper consisted of eight multiple choice questions and four open-answer questions, to be solved within 2 hours. In total 5258 students of 245 secondary schools participated in this first round.

Those 799 students from grade 5 (4, $\leqslant 3$) that scored 15 (13, 11) points or more on the first round (out of a maximum of 36 points) were invited to the second round, which was held in March at ten universities in the country. This round contained five open-answer questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

Those students from grade 5 (4, $\leqslant 3$) that scored 32 (28, 22) points or more on the second round (out of a maximum of 40 points) were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total 149 students were invited. They also received an invitation to some training sessions at the ten universities, in order to prepare them for their participation in the final round.

Out of those 149, in total 142 participated in the final round on 16 September 2011 at Eindhoven University of Technology. This final round contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 50th edition 2011.

The 31 most outstanding candidates of the Dutch Mathematical Olympiad 2011 were invited to an intensive seven-month training programme, consisting of weekly problem sets. Also, the students met twice for a three-day training camp, three times for a day at the university, and finally for a six-day training camp in the beginning of June.

Among the participants of the training programme, there were some extra girls, as this year we would participate in the first European Girls' Mathematical Olympiad (EGMO). In total there were eight girls competing to be in the EGMO team. The team of four girls was selected by a selection test, held on 16 March 2012. They attended the EGMO in Cambridge from 10 until 16 April, and the team returned with two honourable mentions and a bronze medal.

The same selection test was used to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Namur, Belgium, from 20 until 22 April. The Dutch team managed to come first in the country ranking, and received two honourable mentions, two bronze medals, three silver medals and two gold medals.

In June the team for the International Mathematical Olympiad 2012 was selected by two team selection tests on 6 and 9 June 2012. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Buenos Aires, from 29 June until 8 July, together with the team from New Zealand.

For younger students the Junior Mathematical Olympiad was held in October 2011 at the VU University Amsterdam. The students invited to participate in this event were the 30 best students of grade 1, grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with fifteen multiple choice questions and one with ten open-answer questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

## Dutch delegation

The Dutch team for IMO 2012 in Argentina consists of

- Guus Berkelmans (18 y.o., bronze medal at IMO 2010)
- Jeroen Huijben (16 y.o., observer C at IMO 2010, bronze medal at IMO 2011)
- Matthijs Lip (16 y.o.)
- Michelle Sweering (15 y.o.)
- Jeroen Winkel (15 y.o., observer C at IMO 2011)
- Jetze Zoethout (17 y.o., bronze medal at IMO 2011)

We bring as observer C the promising young student

- Peter Gerlagh (15 y.o.)

The team is coached by

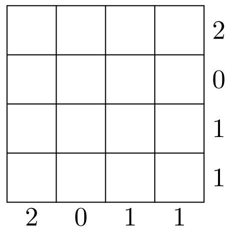
- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Aloysius College The Hague

# First Round, February 2011

## Problems

### A-problems

**A1.** The squares of a 4×4-field are colored black or white. The number next to each row and below each column indicates how many squares in that row or column have to be black.
In how many ways can the field be colored?

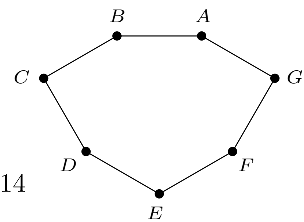



A) 0 B) 1 C) 4 D) 5 E) 8

**A2.** Today is 4 February 2011. This date is written down as 04-02-2011. In this problem we consider the first day from now on, of which the date is written using eight *different* digits.
What is the month of that date?

A) January B) March C) June
D) October E) December

**A3.** A heptagon $ABCDEFG$ is given, all sides of which have length 2. Moreover: $\angle E = 120°$, $\angle C = \angle G = 90°$ and $\angle A = \angle B = \angle D = \angle F$.
What is the area of the heptagon?




A) $10 + 2\sqrt{2}$ B) $8 + 3\sqrt{3}$ C) 14
D) $10 + 2\sqrt{6}$ E) $8 + 3\sqrt{6}$

**A4.** Alice, Brian and Carl have participated in a math contest consisting of 12 problems. Before the contest they were pessimistic and made the following statements.
Alice: "Brian will answer at least two more problems correctly than I will."
Brian: "I will not answer more than five problems correctly."
Carl: "I will at the most answer correctly as many problems as Alice."
Their teacher tried to encourage them by saying: "Together, you will answer more than 18 problems correctly." Afterwards, it transpired that all three students and their teacher had made a wrong prediction.

Who has/have answered the least number of problems correctly?

A) only Alice
B) only Brian
C) only Carl
D) both Alice and Brian
E) you cannot be sure of that

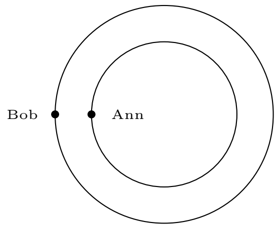
**A5.** Jack wants to write down some of the numbers from 1 to 100 on a piece of paper. He wants to do it in such a way, that no two numbers on the piece of paper will add up to 125.
How many numbers, at most, can Jack write down on his piece op paper?

A) 50 B) 61 C) 62 D) 63 E) 64

**A6.** The number $a = 11\ldots111$ consists of 2011 digits '1'.
What is the remainder of $a$ when divided by 37?

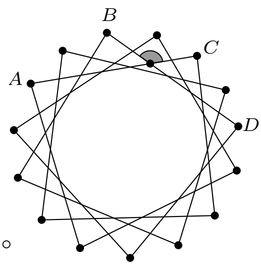
A) 0 B) 1 C) 3 D) 7 E) 11

**A7.** Ann and Bob are sitting in a fairground attraction. They move in circles around the same center and in the same direction. Ann moves around once every 20 seconds, Bob once every 28 seconds. At a certain moment they are at minimum distance from each other (see figure).
How many seconds does it take, from that moment on, until Ann and Bob are at maximum distance from each other?




A) 22.5 B) 35 C) 40 D) 49 E) 70

**A8.** The vertices of a regular 15-gon are connected as in the figure. (Mind you: the sizes in the figure are not entirely correct!)
What is the size of the angle, indicated by the arc, between $AC$ and $BD$?




A) 130° B) 132° C) 135° D) 136° E) 137.5°

## B-problems

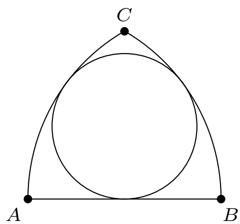
The answer to each B-problem is a number.

**B1.** A number $x$ satisfies: $x = \dfrac{1}{1+x}$. Determine $x - \dfrac{1}{x}$. Simplify your answer as much as possible.

**B2.** An escalator goes up from the first to the second floor of a department store. Dion, while going up the escalator, also walks at a constant pace. Raymond, going in the opposite direction, tries to walk downwards, from the second to the first floor, on the same escalator. He walks at the same pace as Dion. They both take one step of the escalator at a time. Dion arrives at the second floor after exactly 12 steps; Raymond arrives at the first floor after exactly 60 steps.
How many steps would it take Dion to get upstairs if the escalator would stand still?

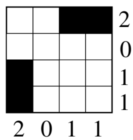
**B3.** Six scouts are on a scouting expedition. They are going to the woods on Saturday, and into the mountains on Sunday. On both days, they have to go in pairs. The scoutmaster wants to group them into pairs for both expeditions in such a way that nobody has the same partner on the second day as on the first day.
In how many ways can he do that?

**B4.** In the figure you see a pointed arch $ABC$ and its inscribed circle. The pointed arch consists of a line segment $AB$ of length 1, a circular arc $BC$ with center $A$ and a circular arc $AC$ with center $B$.
What is the radius of the inscribed circle of this pointed arch?

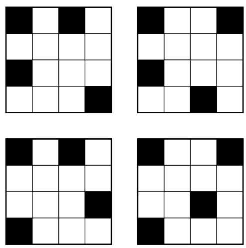



# Solutions

## A-problems



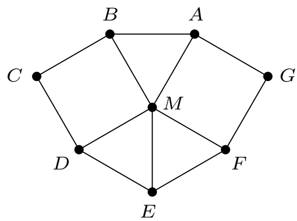

**A1.** D) 5 Notice that all squares in the second row and second column must be white. We consider two cases, depending on the color of the upper left square. If this square is white, then the last two squares in the first row and column must be black. This determines the coloring. See the top figure.
If this square is black, both the first row and first column require exactly one more black square. For each of the resulting $2\times 2 = 4$ choices, there is exactly one solution. Indeed, one row and one column are left that need an additional black square.



Therefore the square at the intersection of this row and column must be colored black, and the remaining squares must be colored white, see bottom four figures.

**A2.** C) June The year of the date we are looking for, starts with a digit 2 or higher. We will look for the first date of which the year starts with digit 2, and all eight digits are different. If such a date exists, we are done. For the month, both 11 (two equal digits) and 12 (digit 2 is already used) can be rejected. Therefore, the month (01 to 10) contains digit 0. This implies that the day starts with digit 1 or 3. In the second case, it's the 31st, since digit 0 is already taken. In both cases, the day contains digit 1. Both digit 0 and 1 being taken, the smallest possible year is 2345. The smallest number we can use for the month is then 06, that is, June. Finally, the day will be the 17th. Observe that the constructed date 17-06-2345 consists of eight different digits, as required.

**A3.** B) $8+3\sqrt{3}$ The heptagon can be partitioned into two squares and three equilateral triangles, all with sides of length 2. We know that each of the squares has an area of 4. Using the Pythagorean theorem, we can compute the height of triangle $ABM$ to be $\sqrt{2^2-1}=\sqrt{3}$. Hence, the area of the triangle equals $\frac{1}{2}\cdot 2\cdot\sqrt{3}=\sqrt{3}$. Summing up the areas of the squares and triangles, we arrive at $2\cdot 4+3\cdot\sqrt{3}=8+3\sqrt{3}$ for the area of the heptagon.



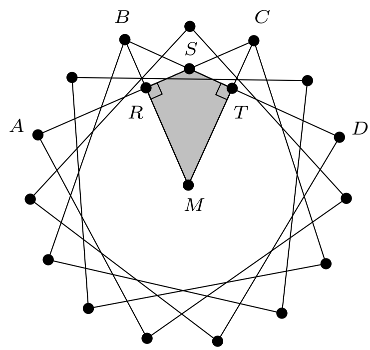

**A4.** A) only Alice As Brian's prediction was wrong, he has at least six correct answers. Alice's prediction was wrong as well, which implies that Brian answered at the most one more question correctly than Alice. Hence, Alice has at least five correct answers. Since Carl made a wrong prediction, he answered more questions correctly than Alice, hence at least six. Alice cannot have more than five correct answers. Indeed, then Carl would have seven correct answers, leading to a total of at least $6+6+7=19$ correct answers, as the teacher (incorrectly) predicted. We can conclude that Alice answered five questions correctly. Since the others have at least six correct answers, Alice is the only one with the smallest number of correct answers.

**A5.** C) 62 Jack can certainly write down 62 numbers, for example: the numbers from 1 to 62 (since $62 + 61 < 125$). More than 62 numbers will not be possible. Indeed, the numbers from 25 to 100 can be partitioned into pairs of sum 125: $25 + 100 = 125$, $26 + 99 = 125$, and so on up to $62 + 63 = 125$. Jack must skip at least one number from each of the 38 pairs. In total, therefore, he can write down no more than $100 - 38 = 62$ of the numbers.

**A6.** B) 1 Using long division to divide $a = 11\cdots11$ (2011 digits) by 37, you will quickly notice that 111 is divisible by 37. This is the fact that we will be using. It implies that the number $1110\cdots0$ is divisible by 37, regardless of the number of trailing zeros. In particular, the following numbers are divisible by 37: $1110\cdots0$ (2008 zeros), $1110\cdots0$ (2005 zeros), $1110\cdots0$ (2002 zeros), and so on up to 1110 (1 zero). The sum of these numbers equals $1\ldots10$ (2010 digits 1), which is again divisible by 37. In conclusion, the remainder of $a$ when divided by 37 is 1, since $a - 1$ is divisible by 37.

**A7.** C) 35 After 140 seconds, Ann has made 7 rounds and Bob only 5. At that moment, Ann leads by two full rounds. Hence after only $\frac{140}{4} = 35$ seconds, Anne leads by half a round. That is exactly the first moment she and Bob are at maximal distance from each other.

**A8.** B) 132° Denote the center of the 15-gon by $M$ and the intersection of $AC$ and $BD$ by $S$ (see figure). In quadrilateral $MRST$, we see that $\angle MRS = 90°$ and $\angle STM = 90°$. Furthermore, we see that $\angle TMR = \frac{2}{15} \cdot 360° = 48°$. As the angles of a quadrilateral sum to 360°, we find: $\angle RST = 360° - 2 \cdot 90° - 48° = 132°$. Observe that $\angle RST$ and $\angle BSC$ are opposite angles. Hence the sought-after angle also equals 132°.



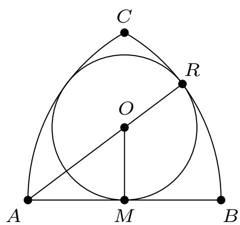
## B-problems

**B1.** $-1$ We are given that $x = \frac{1}{1+x}$. Clearly, $x \neq 0$, since $0 \neq \frac{1}{1}$. Hence, on both sides of the equation, we may flip the numerator and denominator of the fraction. This results in: $\frac{1}{x} = 1 + x$. Combining both formulas, we obtain $x - \frac{1}{x} = x - (1 + x) = -1$.

**B2.** $20$ Consider three escalators in a row: the first one going up, the second standing still, and the third going down. If Dion walks up the first escalator, he arrives at the top after exactly 12 steps. Raymond, walking up the third escalator, takes 60 steps to reach the top and reaches only $\frac{1}{5}$ of the escalator after 12 steps. A third person, say Julian, takes the second escalator and walks at the same pace as Dion and Raymond. After 12 steps, he will be positioned exactly in between Dion and Julian, at $(\frac{5}{5} + \frac{1}{5})/2 = \frac{3}{5}$ of the escalator. Therefore, he will need $\frac{5}{3} \cdot 12 = 20$ steps to reach the top.

**B3.** $120$ Let's call the eldest scout $A$. There are 5 possibilities for finding him a partner $B$ for the first day. Then, there are 4 possible partners $C$ for $B$ on the second day, because he cannot be paired with $A$ twice. Now for $C$, there are 3 possible partners $D$ on the first day, since he cannot go with $B$ again, and $A$ is already paired. For $D$, there are now 2 possible partners $E$ on the second day, since $B$ and $C$ are already paired, and $A$ cannot be his partner because that would leave two scouts that are forced to form a pair on both days. Finally, there is one scout left. He has no choice but to team op with $E$ on the first day, and with $A$ on the second. In total there are $5 \times 4 \times 3 \times 2 = 120$ possibilities.

**B4.** $\frac{3}{8}$ Consider the inscribed circle. We denote its center by $O$ and its radius by $r$. The points where the circle is tangent to $AB$ and $BC$ are denoted by $M$ and $R$, respectively.
Since $A$, $O$ and $R$ are on a line, we have: $|AO| = |AR| - |OR| = 1 - r$. We also know that $|OM| = r$ and $|AM| = \frac{1}{2}|AB| = \frac{1}{2}$. Using the Pythagorean theorem, we find $|AM|^2 + |OM|^2 = |AO|^2$, and hence $\frac{1}{4} + r^2 = (1 - r)^2 = r^2 - 2r + 1$. This implies that $2r = \frac{3}{4}$ and therefore $r = \frac{3}{8}$.



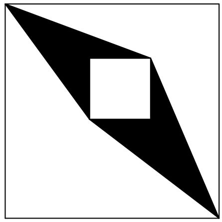
# Second Round, March 2011

## Problems

### B-problems

The answer to each B-problem is a number.

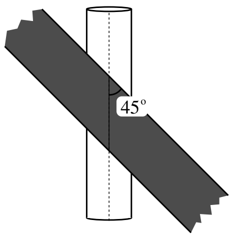
**B1.** At a gala, a number of pairs (consisting of one man and one woman) are dancing, in such a way that $\frac{2}{3}$ of the women present is dancing with $\frac{3}{5}$ of the men present.
What part of those present at the gala is dancing?



**B2.** A square with edges of length 2 is inside of a square with edges of length 7. The edges of the smaller square are parallel to the ones of the larger one.
What is the area of the black-coloured part?

**B3.** In a classroom, there are 23 students who chose to learn a foreign language, namely German and French. Of those 23 students, 10 are girls, and 11 of those 23 students have chosen French as their foreign language. The number of girls that have chosen French, plus the number of boys that have chosen German, is equal to 16.
What is the number of girls that have chosen French?

**B4.** We have a deck of 10.000 cards, numbered from 1 to 10.000. A step consists of removing all the cards that has a square on it, and then renumbering the remaining cards, starting from 1, in a successive way.
What is the number of steps needed to remove all but one card?




**B5.** We put a grey ribbon over a cylindrical white pole, under an angle of 45 degrees. The ribbon is then wound tightly around the pole (without creases). In this way, we get a grey spiral around the pole. Between the grey spiral, a white spiral runs around the pole; that is the part of the pole that is not covered by the ribbon. The radius of the pole is 2 cm.
It turns out that the white and grey spirals have the same widths. What is the width of the ribbon?

## C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

**C1.** Determine all triples $(a, b, c)$ of positive integers with the following properties:

- we have $a < b < c$, and $a$, $b$ and $c$ are three successive odd integers;
- the number $a^2 + b^2 + c^2$ consists of four equal digits.

**C2.** Thirty students participate in a mathematical competition with sixteen questions. They have to answer each question with a number. If a student answers a question correctly within a minute, he gets 10 points for that question. If a student answers a question correctly, but not within one minute, then he gets 5 points for that question. And if a student answers a question incorrectly, he gets no points at all for that question.
After the competition has ended, it turns out that from all the 480 answers that were given, more than half was correct and given within a minute. The number of answers that were correct, but not given within a minute, turns out to be equal to the number of incorrect answers.
Show that there are two students with the same total score.

# Solutions

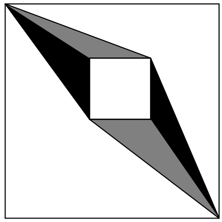
## B-problems

**B1.** $\frac{12}{19}$ Let $w$ be the number of women present, and let $m$ be the number of men present. The problem tells us that $\frac{2}{3}w = \frac{3}{5}m$ and hence that $w = \frac{9}{10}m$. The number of people dancing, is exactly twice the number of men dancing, namely $\frac{6}{5}m$.

The number of people present is of course $m + w = m + \frac{9}{10}m = \frac{19}{10}m$. So it follows that the part of those present that is dancing, is equal to

$$\frac{\frac{6}{5}m}{\frac{19}{10}m} = \frac{6}{5} \cdot \frac{10}{19} = \frac{12}{19}.$$

**B2.** 10 We split the black part into four triangles, of which we coloured two of them gray. The two gray triangles both have base 2, and their combined height is $7 - 2 = 5$, namely the height of the larger square minus the height of the smaller square. Hence the area of the two grey triangles together is equal to $\frac{1}{2} \cdot 2 \cdot 5 = 5$. The same holds for the two black triangles. It follows that the combined area is $5 + 5 = 10$.



**B3.** 7 There are 23 students in total. From what's given, it follows that:

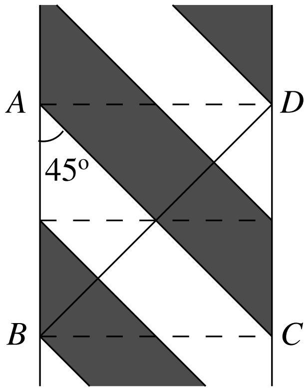
$$\begin{aligned}
16 + 11 + 10 &= (\text{girls with French} + \text{boys with German}) \\
&\quad + \text{everyone with French} + \text{all girls} \\
&= (\text{girls with French} + \text{boys with German}) \\
&\quad + (\text{girls with French} + \text{boys with French}) \\
&\quad + (\text{girls with French} + \text{girls with German}) \\
&= 3 \times \text{girls with French} + \text{boys with German} \\
&\quad + \text{boys with French} + \text{girls with German} \\
&= 2 \times \text{girls with French} + 23.
\end{aligned}$$

So the total number of girls that have chosen French is equal to $\frac{16+11+10-23}{2} = \frac{14}{2} = 7$.

**B4.** 198 On the first step, we remove the cards numbered by $1^2$, $2^2$, $3^2$, ..., $100^2$. Then 9900 cards remain. Since $99^2 \leqslant 9900 < 100^2$, we remove $1^2$, $2^2$, ..., $99^2$ in the second step. After that, $9900 - 99 = 9801 = 99^2$ cards are left, which is a square.

In general, if we start with $n^2$ cards, with $n \geqslant 2$, we remove $n$ cards in the first step, after which $n^2 - n$ cards remain. Since $(n-1)^2 = n^2 - 2n + 1 \leqslant n^2 - n < n^2$, we remove $n - 1$ cards in the second step. Then exactly $(n^2 - n) - (n - 1) = (n - 1)^2$ are left. So in two steps we can reduce the number of cards from $n^2$ to $(n - 1)^2$. It follows that we need $2 \cdot 99 = 198$ steps to remove all but one of the cards when we start with $100^2$ cards.

**B5.** $\pi\sqrt{2}$ cm Imagine the pole as a paper cylinder. Cut it open along its length, then unroll it, to get a rectangular strip of paper. So points $A$ and $D$ correspond to the same point on the cylinder, just like points $B$ and $C$. The width of the strip is equal to the perimeter of the cylinder, so $|AD| = |BC| = 2\pi \cdot 2$ cm $= 4\pi$ cm.




Note that the grey ribbon forms a 45° angle with the cutting line, $ABCD$ is a square. The length of the diagonal $BD$ is equal to $\sqrt{2} \cdot 4\pi$ cm and also equal to four times the width of the grey ribbon, since the white and grey stripes have the same width. It follows that the grey ribbon has width $\pi\sqrt{2}$ cm.

## C-problems

**C1.** Since $a$, $b$ and $c$ are three successive positive odd integers, we can write: $a = 2n - 1$, $b = 2n + 1$ and $c = 2n + 3$, with $n$ a positive integer.
A calculation then gives:

$$\begin{aligned} a^2 + b^2 + c^2 &= (2n-1)^2 + (2n+1)^2 + (2n+3)^2 \\ &= (4n^2 - 4n + 1) + (4n^2 + 4n + 1) + (4n^2 + 12n + 9) \\ &= 12n^2 + 12n + 11. \end{aligned}$$

This needs to be equal to an integer that consists of four digits $p$. Hence the integer $12n^2 + 12n$ consists of four digits, of which the first two are equal to $p$, and the last two are equal to $p - 1$. Since $12n^2 + 12n$ is divisible by 2, $p - 1$ has to be even. So we have the following possibilities for $12n^2 + 12n$; 1100, 3322, 5544, 7766 and 9988. This integer also must be divisible by 3, so the only integer remaining is 5544, so $n^2 + n = \frac{5544}{12} = 462$. We can rewrite this as $n^2 + n - 462 = 0$. Factorizing this quadratic equation then gives: $(n - 21)(n + 22) = 0$. Since $n$ is a positive integer, the only solution is $n = 21$. So the only triple satisfying the given properties is $(a, b, c) = (41, 43, 45)$.

**C2.** Note that the possible scores are multiples of 5. The lowest score a student can get is 0, and the highest score is $16 \cdot 10 = 160$. Now suppose that there are no two students with the same score. Then the combined score of the students is at most $160 + 155 + 150 + \cdots + 15 = \frac{1}{2} \cdot 175 \cdot 30 = 2625$. We'll derive a contradiction from this.

Let $A$ be the combined number of correct answers that were given within one minute, $B$ be the combined number of correct answers that were not given within a minute, and $C$ be the combined number of incorrect answers. The students answered $16 \cdot 30 = 480$ questions together, so $A+B+C = 480$. More than half of the questions was answered correctly within one minute, so $A > 240$. Also note that $B = C$, so $B = C = \frac{480-A}{2}$. We now can express the combined score in $A$. This is equal to:

$$10 \cdot A + 5 \cdot B + 0 \cdot C = 10 \cdot A + 5 \cdot \frac{480 - A}{2} = \frac{15}{2} A + 1200.$$

Since $A > 240$, the combined scores of the students is greater than $\frac{15}{2} \cdot 240 + 1200 = 3000$. But from the assumption that no two students have the same score, we deduced that the combined score was at most 2625. This is a contradiction. We deduce that this assumption was wrong, so that there are two students with the same score.

# Final Round, September 2011

## Problems

For these problems not only the answer is important; you also have to describe the way you solved the problem.

**1.** Determine all triples of positive integers $(a, b, n)$ that satisfy the following equation:

$$a! + b! = 2^n.$$

*Notation:* $k! = 1\times 2\times\cdots\times k$, for example: $1! = 1$, and $4! = 1\times 2\times 3\times 4 = 24$.

**2.** Let $ABC$ be a triangle. Points $P$ and $Q$ lie on side $BC$ and satisfy $|BP| = |PQ| = |QC| = \frac{1}{3}|BC|$. Points $R$ and $S$ lie on side $CA$ and satisfy $|CR| = |RS| = |SA| = \frac{1}{3}|CA|$. Finally, points $T$ and $U$ lie on side $AB$ and satisfy $|AT| = |TU| = |UB| = \frac{1}{3}|AB|$. Points $P, Q, R, S, T$ and $U$ turn out to lie on a common circle.
Prove that $ABC$ is an equilateral triangle.

**3.** In a tournament among six teams, every team plays against each other team exactly once. When a team wins, it receives 3 points and the losing team receives 0 points. If the game is a draw, the two teams receive 1 point each.
Can the final scores of the six teams be six consecutive numbers $a, a+1, \ldots, a+5$? If so, determine all values of $a$ for which this is possible.

**4.** Determine all pairs of positive real numbers $(a, b)$ with $a > b$ that satisfy the following equations:

$$a\sqrt{a} + b\sqrt{b} = 134 \qquad \text{and} \qquad a\sqrt{b} + b\sqrt{a} = 126.$$

**5.** The number devil has coloured the integer numbers: every integer is coloured either black or white. The number 1 is coloured white. For every two white numbers $a$ and $b$ ($a$ and $b$ are allowed to be equal) the numbers $a - b$ and $a + b$ have different colours.
Prove that 2011 is coloured white.

## Solutions

**1.** Since $a$ and $b$ play the same role in the equation $a! + b! = 2^n$, we will assume for simplicity that $a \leqslant b$. The solutions for which $a > b$ are found by interchanging $a$ and $b$. We will consider the possible values of $a$.

**Case $a \geqslant 3$:** Since $3 \leqslant a \leqslant b$, both $a!$ and $b!$ are divisible by 3. Hence $a! + b!$ is divisible by 3 as well. Because $2^n$ is not divisible by 3 for any value of $n$, we find no solutions in this case.

**Case $a = 1$:** The number $b$ must satisfy $b! = 2^n - 1$. This implies that $b!$ is *odd*, because $2^n$ is *even* (recall that $n \geqslant 1$). Since $b!$ is divisible by 2 for all $b \geqslant 2$, we must have $b = 1$. We find that $1! = 2^n - 1$, which implies that $n = 1$. The single solution in the case is therefore $(a, b, n) = (1, 1, 1)$.

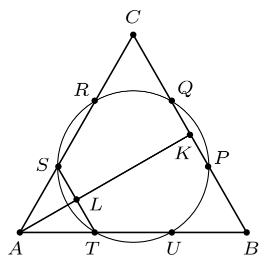
**Case $a = 2$:** There are no solutions for $b \geqslant 4$. Indeed, since $b!$ would then be divisible by 4, $2^n = b! + 2$ would *not* be divisible by 4, which implies that $2^n = 2$. However, this contradicts the fact that $2^n = b! + 2 \geqslant 24 + 2$.

For $b = 2$, we find $2^n = 2+2 = 4$. Hence $n = 2$ and $(a, b, n) = (2, 2, 2)$ is the only solution.

For $b = 3$, we find $2^n = 2 + 6 = 8$. Hence $n = 3$ and $(a, b, n) = (2, 3, 3)$ is the only solution. By interchanging $a$ and $b$, we obtain the additional solution $(a, b, n) = (3, 2, 3)$.

In all, there are four solutions: $(1, 1, 1)$, $(2, 2, 2)$, $(2, 3, 3)$ and $(3, 2, 3)$. $\square$

**2.** Denote by $K$ the midpoint of $PQ$. Then $K$ is also the midpoint of $BC$, and $AK$ is a median of triangle $ABC$. We denote by $L$ the intersection of $AK$ and $ST$.




Triangles $AST$ and $ACB$ are similar (sas), because $\angle CAB = \angle SAT$ and $\frac{|CA|}{|SA|} = 3 = \frac{|BA|}{|TA|}$. This implies that $ST$ and $CB$ are parallel lines (equal corresponding angles).

Triangles $ASL$ and $ACK$ are similar (aa), because $\angle SAL = \angle CAK$ and $\angle LSA = \angle TSA = \angle BCA = \angle KCA$. Hence $\frac{|CK|}{|SL|} = \frac{|CA|}{|SA|} = 3$. This implies that $L$ is the midpoint of $ST$, because $\frac{|SL|}{|ST|} = \frac{3 \cdot |SL|}{3 \cdot |ST|} = \frac{|CK|}{|CB|} = \frac{1}{2}$.

Consider the center $M$ of the circle through $P$, $Q$, $R$, $S$, $T$ and $U$. It is incident to both the perpendicular bisector of $PQ$, and that of $ST$.

However, since $PQ$ and $ST$ are parallel, the two perpendicular bisectors must coincide: they are the same line. This line is incident to $L$ and $K$, and is therefore equal to line $AK$, which shows that $AK \perp BC$.

It follows that $|AC| = |AB|$, because $AK$ is the perpendicular bisector of $BC$.

In a similar fashion, one can show that $|AC| = |BC|$, concluding the proof that triangle $ABC$ is equilateral. $\square$

**3.** In all, 15 matches are played. In each match, the two teams together earn 2 or 3 points. The sum of the final scores is therefore an integer between $15 \cdot 2 = 30$ (all matches end in a draw) and $15 \cdot 3 = 45$ (no match is a draw).

On the other hand, the sum of the six scores equals $a+(a+1)+\cdots+(a+5) = 15 + 6a$. Hence $30 \leqslant 15 + 6a \leqslant 45$, which shows that $3 \leqslant a \leqslant 5$. We will prove that $a = 4$ is the only possibility.

First consider the case $a = 5$. The sum of the scores equals $15 + 30 = 45$, so no match ends in a draw. Because in every match the teams earn either 0 or 3 points, every team's score is divisible by 3. Therefore, the scores cannot be six consecutive numbers.

Next, consider the case $a = 3$. The scores sum up to $3+4+5+6+7+8 = 33$. The two teams scoring 6 and 7 points must both have won at least one out of the five matches they played.

The team scoring 8 points must have won at least two matches, because $3 + 1 + 1 + 1 + 1 = 7 < 8$. Hence at least 4 matches did not end in a draw, which implies that the sum of the scores is at least $4 \cdot 3 + 11 \cdot 2 = 34$. But we have already see that this sum equals 33, a contradiction.

Finally, we will show that $a = 4$ is possible. The table depicts a possible outcome in which teams $A$ to $F$ have scores 4 to 9. The rightmost column shows the total scores of the six teams.

| | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | |
|---|---|---|---|---|---|---|---|
| $A$ | - | 3 | 1 | 0 | 0 | 0 | 4 |
| $B$ | 0 | - | 1 | 0 | 3 | 1 | 5 |
| $C$ | 1 | 1 | - | 3 | 0 | 1 | 6 |
| $D$ | 3 | 3 | 0 | - | 1 | 0 | 7 |
| $E$ | 3 | 0 | 3 | 1 | - | 1 | 8 |
| $F$ | 3 | 1 | 1 | 3 | 1 | - | 9 |

$\square$

**4.** For convenience, write $y = \sqrt{a}$ and $z = \sqrt{b}$. The equations transform to

$$y^3 + z^3 = 134 \qquad \text{and} \qquad y^2z + yz^2 = 126.$$

Combining these two equations in a handy way, we find

$$(y+z)^3 = (y^3 + z^3) + 3(y^2z + yz^2) = 134 + 3 \cdot 126 = 512 = 8^3.$$

This immediately implies that $y + z = 8$.

Rewrite the first equation as follows: $(y + z)yz = y^2z + yz^2 = 126$. Since $y + z = 8$, we see that $yz = \frac{126}{8} = \frac{63}{4}$.

From $y + z = 8$ and $yz = \frac{63}{4}$, we can determine $y$ and $z$ by solving a quadratic equation: $y$ and $z$ are precisely the roots of the equation $x^2 - 8x + \frac{63}{4} = 0$. The two solutions are $\frac{8 \pm \sqrt{64 - 4 \cdot \frac{63}{4}}}{2}$, that is $\frac{9}{2}$ and $\frac{7}{2}$.

Since $a > b$, also $y > z$ holds. Hence $y = \frac{9}{2}$ and $z = \frac{7}{2}$. We therefore find that $(a, b) = (\frac{81}{4}, \frac{49}{4})$.

Because $(a, b) = (\frac{81}{4}, \frac{49}{4})$ satisfies the given equations, as required, we conclude that this is the (only) solution. $\square$

**5.** We are give that 1 is white. Hence 0 is black, because otherwise $1 = 1 - 0$ and $1 = 1 + 0$ would have different colours. The number 2 is white, because $0 = 1 - 1$ (black) and $2 = 1 + 1$ have different colours.

By induction on $k$, we show that the following claim holds for every $k \geqslant 0$:

$$3k \text{ is black, } 3k + 1 \text{ and } 3k + 2 \text{ are white.}$$

We have just shown the base case $k = 0$. Assume that the claim holds true for $k = \ell$.

Since 1 is white, and $3\ell + 2$ is white by the induction hypothesis, the numbers $(3\ell + 2) - 1 = 3\ell + 1$ and $(3\ell + 2) + 1 = 3(\ell + 1)$ have different colours. As $3\ell + 1$ is white by the induction hypothesis, $3(\ell + 1)$ must be black.

Since 2 and $3\ell + 2$ are both white, the numbers $(3\ell + 2) + 2 = 3(\ell + 1) + 1$ and $(3\ell + 2) - 2 = 3\ell$ must have different colours. As $3\ell$ is black by the induction hypothesis, $3(\ell + 1) + 1$ must be white.

Since $3(\ell + 1) + 1$ and 1 are both white, the numbers $3(\ell + 1) + 1 + 1 = 3(\ell + 1) + 2$ and $3(\ell + 1)$ have different colours. We already know that $3(\ell + 1)$ is black, so $3(\ell + 1) + 2$ must be white.

This proves the claim for $k = \ell + 1$.

Because $2011 = 3 \cdot 670 + 1$, this shows that 2011 is white. $\square$

# BxMO/EGMO Team Selection Test, March 2012

## Problems

**1.** Do there exist quadratic polynomials $P(x)$ and $Q(x)$ with real coefficients such that the polynomial $P(Q(x))$ has precisely the zeros $x = 2, x = 3, x = 5$ and $x = 7$?

**2.** Let $\triangle ABC$ be a triangle and let $X$ be a point in the interior of the triangle. The second intersection points of the lines $XA, XB$ and $XC$ with the circumcircle of $\triangle ABC$ are $P, Q$ and $R$. Let $U$ be a point on the ray $XP$ (these are the points on the line $XP$ such that $P$ and $U$ lie on the same side of $X$). The line through $U$ parallel to $AB$ intersects $BQ$ in $V$. The line through $U$ parallel to $AC$ intersects $CR$ in $W$.
Prove that $Q$, $R$, $V$, and $W$ lie on a circle.

**3.** Find all pairs of positive integers $(x, y)$ for which

$$x^3 + y^3 = 4(x^2y + xy^2 - 5).$$

**4.** Let $ABCD$ a convex quadrilateral (this means that all interior angles are smaller than $180°$), such that there exist a point $M$ on line segment $AB$ and a point $N$ on line segment $BC$ having the property that $AN$ cuts the quadrilateral in two parts of equal area, and such that the same property holds for $CM$.
Prove that $MN$ cuts the diagonal $BD$ in two segments of equal length.

**5.** Let $A$ be a set of positive integers having the following property: for each positive integer $n$ exactly one of the three numbers $n, 2n$ and $3n$ is an element of $A$. Furthermore, it is given that $2 \in A$. Prove that $13824 \notin A$.

# Solutions

**1.** Suppose that such polynomials exist and write $Q(x) = ax^2 + bx + c$. If we evaluate $Q$ in 2, 3, 5, and 7, we must get exactly the (at most) two zeros of $P$. Since $Q(x)$ attains each value at most two times (because $Q$ is quadratic) we get two different values exactly two times.

Now suppose that $Q(n) = Q(m)$ for different numbers $m$ and $n$. Then $an^2 + bn + c = am^2 + bm + c$, hence $a(n^2 - m^2) = b(m - n)$, hence $a(n+m)(n-m) = -b(n-m)$. As $m - n \neq 0$, this yields $a(n+m) = -b$, or equivalently, $n + m = \frac{-b}{a}$.

We know that 2, 3, 5 and 7 split in two pairs $(m, n)$ and $(k, l)$ such that $Q(m) = Q(n)$ and $Q(k) = Q(l)$. Therefore, it holds that $m + n = \frac{-b}{a} = k + l$. We have to split the four numbers 2, 3, 5 and 7 in two pairs having the same sum. However, this is impossible, since $2+3+5+7 = 17$ is odd. We conclude that there are no polynomials $P$ and $Q$ having the required properties. $\square$

**2.** As $AB$ and $UV$ are parallel, it holds that $\Delta ABC \sim \Delta UVX$ (AA). Analogously, it holds that $\Delta ACX \sim \Delta UWX$. These similarities yield that

$$\frac{|XA|}{|XB|} = \frac{|XU|}{|XV|} \qquad \text{and} \qquad \frac{|XA|}{|XC|} = \frac{|XU|}{|XW|}.$$

This yields that

$$\frac{|XB|}{|XC|} = \frac{|XB|}{|XA|}\frac{|XA|}{|XC|} = \frac{|XV|}{|XU|}\frac{|XU|}{|XW|} = \frac{|XV|}{|XW|}.$$

The power of a point theorem for the point $X$ and the cyclic quadrilateral $BCQR$ gives that

$$|XC| \cdot |XR| = |XB| \cdot |XQ| = \frac{|XB|}{|XC|} \cdot |XC| \cdot |XQ| = \frac{|XV|}{|XW|} \cdot |XC| \cdot |XQ|,$$

hence

$$|XW| \cdot |XR| = |XV| \cdot |XQ|.$$

From the given configuration ($X$ in the interior of the triangle, $U$ and $P$ on the line $XP$ on the same side of $X$) follows that $R$ and $W$ lie on the same side of $X$ on the line $XC$, and also that $V$ and $Q$ lie on the same side of $X$ on the line $XB$. Hence, the following equality also holds for the directed distance: $XW \cdot XR = XV \cdot XQ$. The power of a point theorem then yields that $W, R, V$ and $Q$ lie on a circle. $\square$

**3.** It holds that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, hence the given equality yields

$$\begin{aligned}(x+y)^3 &= x^3 + y^3 + 3x^2y + 3xy^2 \\ &= 4(x^2y + xy^2 - 5) + 3x^2y + 3xy^2 \\ &= 7x^2y + 7xy^2 - 20 \\ &= 7xy(x+y) - 20.\end{aligned}$$

Now $x+y$ is a divisor of 20, because $x+y$ is a divisor of $(x+y)^3$ and $7xy(x+y)$. As $x+y \geqslant 2$, the possible values for $x+y$ are $2, 4, 5, 10, 20$. Considering the equality $(x+y)^3 - 7xy(x+y) - 20$ modulo 7 yields

$$(x+y)^3 \equiv -20 \equiv 1 \mod 7.$$

For each of the possible values for $x+y$ we check whether its third power is congruent to 1 modulo 7. It holds that $5^3 \equiv -1 \mod 7$, hence $x+y=5$ is impossible. It holds that $10^3 \equiv -1 \mod 7$, hence $x+y=10$ is also impossible. It holds that $20^3 \equiv -1 \mod 7$, hence $x+y=20$ is also impossible. We are left with the possibilities $x+y=2$ and $x+y=4$.
If $x+y=2$, then $x=y=1$ and then the left hand site is positive and the right hand side is negative, hence also $x+y=2$ is impossible. If $x+y=4$, then $(x,y)$ is equal to $(1,3), (2,2)$ or $(3,1)$. By substituting this into the equality we find that only $(1,3)$ and $(3,1)$ are solutions. □

**4.** For a polygon $\mathcal{P}$, denote by $O(\mathcal{P})$ the area of $\mathcal{P}$. Let $T$ be the midpoint of $BD$. Now triangles $CDT$ and $CBT$ have equal length bases, namely $|DT| = |BT|$, and their altitudes have the same length, hence their areas are equal. In the same way it holds that $O(ADT) = O(ABT)$. Hence, $O(ATCD) = \frac{1}{2}O(ABCD) = O(ANCD)$. Remark that $T$ cannot lie in the interior of triangle $ACD$, because then $O(ATCD) < O(ACD)$ would hold, while it holds that $O(ANCD) > O(ACD)$. Therefore,

$$O(ATC) = O(ATCD) - O(ACD) = O(ANCD) - O(ACD) = O(ANC).$$

Triangles $ATC$ and $ANC$ have the same base $AC$, hence their altitudes have equal lengths. This means that the line $NT$ is parallel to the base $AC$.
Analogously, we show that $MT$ is parallel to $AC$. Hence, $NT \parallel MT$ and this yields that $M$, $N$ and $T$ lie on a line. Hence, the midpoint of $BD$ lies on the line $MN$. □

**5.** We will prove the following two assertions:

(i) If $m \in A$ and $2 \mid m$, then $6m \in A$.

(ii) If $m \in A$ and $3 \mid m$, then $\frac{4}{3}m \in A$.

First we prove assertion (i). Suppose that $m \in A$ and $2 \mid m$. By choosing $n = \frac{m}{2}$, we find that $\frac{m}{2}$ and $\frac{3}{2}m$ do not lie in $A$. By choosing $n = m$, we find that $2m$ and $3m$ do niet lie in $A$. If we now consider $n = \frac{3}{2}m$, then as $2n = 3m$ and $n = \frac{3}{2}m$ do not lie in $A$, hence $3n = \frac{9}{2}m$ lies in $A$. Using $n = \frac{9}{2}m$, we find that $9m \notin A$. Finally, we consider $n = 3m$: we know that $3m \notin A$ and $9m \notin A$, hence $6m \in A$.
Now we prove assertion (ii). Suppose that $m \in A$ and $3 \mid m$. By choosing $n = \frac{m}{3}$, we find that $\frac{m}{3}$ and $\frac{2}{3}m$ do not lie in $A$. By choosing $n = m$, we find that $2m$ does not lie in $A$. If we now choose $n = \frac{2}{3}m$, we know that $n$ and $3n$ do not lie in $A$, hence $2n = \frac{4}{3}m$ does lie in $A$. This proves assertion (ii).
We know that $2 \in A$. By repeatedly applying assertions (i) and (ii), we show that the following numbers all lie in $A$:

$$2 \overset{\text{(i)}}{\to} 2^2 \cdot 3 \overset{\text{(i)}}{\to} 2^3 \cdot 3^2 \overset{\text{(i)}}{\to} 2^4 \cdot 3^3 \overset{\text{(i)}}{\to} 2^5 \cdot 3^4 \overset{\text{(ii)}}{\to} 2^7 \cdot 3^3 \overset{\text{(ii)}}{\to} 2^9 \cdot 3^2.$$

As $2^9 \cdot 3^2 \in A$, it holds that $2^9 \cdot 3^3 \notin A$. Since $13824 = 2^9 \cdot 3^3$, this is what we had to prove. $\square$

# IMO Team Selection Test 1, June 2012

## Problems

**1.** Let $I$ be the incentre of triangle $ABC$. A line through $I$ intersects the interior of line segment $AB$ in $M$ and the interior of line segment $BC$ in $N$. We assume that $BMN$ is an acute triangle. Let $K$ and $L$ be points on line segment $AC$ such that $\angle BMI = \angle ILA$ and $\angle BNI = \angle IKC$.
Prove that $|AM| + |KL| + |CN| = |AC|$.

**2.** Let $a$, $b$, $c$ and $d$ be positive real numbers. Prove that

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geqslant 0.$$

**3.** Determine all positive integers that cannot be written as $\frac{a}{b} + \frac{a+1}{b+1}$ where $a$ and $b$ are positive integers.

**4.** Let $n$ be a positive integer divisible by 4. We consider the permutations $(a_1, a_2, \dots, a_n)$ of $(1, 2, \dots, n)$ having the following property: for each $j$ we have $a_i + j = n + 1$ where $i = a_j$. Prove that there are exactly $\frac{(\frac{1}{2}n)!}{(\frac{1}{4}n)!}$ such permutations.

**5.** Let $\Gamma$ be the circumcircle of the acute triangle $ABC$. The angle bisector of angle $ABC$ intersects $AC$ in the point $B_1$ and the short arc $AC$ of $\Gamma$ in the point $P$. The line through $B_1$ perpendicular to $BC$ intersects the short arc $BC$ of $\Gamma$ in $K$. The line through $B$ perpendicular to $AK$ intersects $AC$ in $L$.
Prove that $K$, $L$ and $P$ lie on a line.

# Solutions

**1.** Let $D$, $E$ and $F$ be the orthogonal projections of $I$ on respectively $BC$, $CA$ and $AB$. It holds that $N$ lies between $C$ and $D$: since if $N$ would lie between $D$ and $B$, then $\angle BNI$ would be greater than $\angle BDI = 90°$, but it is given that $\triangle BMN$ is acute. Hence, $N$ lies between $C$ and $D$. Analogously, $M$ lies between $A$ and $F$. Furthermore, $L$ cannot lie between $A$ and $E$, because in that case, $\angle ILA > 90°$ would hold, while we have $\angle ILA < \angle BMI < 90°$. Hence, $L$ lies between $E$ and $C$. Analogously, $K$ lies between $A$ and $E$. Altogether, $E$ lies between $K$ and $L$.
It holds that

$$|AC| = |AE| + |CE| = |AF| + |CD| = |AM| + |MF| + |CN| + |ND|,$$

where the second equality holds because the segments on the tangent lines to the incircle have equal lengths.
Furthermore, we have $\angle IKE = \angle IKC = \angle BNI = \angle DNI$ and $\angle KEI = 90° = \angle IDN$, hence $\triangle IKE \sim \triangle IND$ (AA). Since line segments $EI$ and $DI$ both are radii of the incircle, they have equal lengths, hence we even have $\triangle IKE \cong \triangle IND$. This yields that $|EK| = |ND|$.
Analogously, we find that $|EL| = |MF|$. Altogether, we have

$$\begin{aligned}|AC| &= |AM| + |MF| + |ND| + |CN| \\ &= |AM| + |EL| + |EK| + |CN| = |AM| + |KL| + |CN|. \qquad \square\end{aligned}$$

**2.** It holds that

$$\frac{a-b}{b+c} = \frac{a-b+b+c}{b+c} - 1 = \frac{a+c}{b+c} - 1.$$

By applying the equality also to the other three fractions and adding 4 to both sides, we find that it is sufficient to prove that:

$$\text{LHS} := \frac{a+c}{b+c} + \frac{b+d}{c+d} + \frac{c+a}{d+a} + \frac{d+b}{a+b} \geqslant 4. \tag{1}$$

Next, we apply the inequality of the harmonic and geometric mean to the two positive numbers $b+c$ and $d+a$:

$$\frac{2}{\frac{1}{b+c} + \frac{1}{d+a}} \leqslant \frac{(b+c)+(d+a)}{2},$$

hence,

$$\frac{1}{b+c} + \frac{1}{d+a} \geqslant \frac{4}{a+b+c+d}.$$

Analogously, it holds that

$$\frac{1}{c+d} + \frac{1}{a+b} \geqslant \frac{4}{a+b+c+d}.$$

Using this, we find the following inequality for the left-hand side of (1):

$$\begin{aligned}
\text{LHS} &= (a+c)\left(\frac{1}{b+c} + \frac{1}{d+a}\right) + (b+d)\left(\frac{1}{c+d} + \frac{1}{a+b}\right) \\
&\geqslant (a+c)\cdot\frac{4}{a+b+c+d} + (b+d)\cdot\frac{4}{a+b+c+d} \\
&= 4\cdot\frac{(a+c)+(b+d)}{a+b+c+d} \\
&= 4.
\end{aligned}$$

This proves (1). □

**3.** It holds that

$$\frac{a}{b} + \frac{a+1}{b+1} = \frac{2ab+a+b}{b(b+1)}.$$

Next, suppose that this is equal to an integer $n$. Then we have $b \mid 2ab+a+b$ and $b+1 \mid 2ab+a+b$. The former yields that $b \mid a$ and hence $b \mid a-b$. The latter yields $b+1 \mid (2ab+a+b)-(b+1)\cdot 2a = -a+b$ and hence also $b+1 \mid a-b$. Since the gcd of $b$ and $b+1$ is equal to 1, we may conclude that $b(b+1) \mid a-b$. Hence, we can write $a$ as $a = b(b+1)\cdot k + b$. Since $a$ is positive, $k$ must be a non-negative integer. Substitution yields

$$\begin{aligned}
n &= \frac{2ab+a+b}{b(b+1)} \\
&= \frac{2\cdot\big(b(b+1)\cdot k+b\big)\cdot b + \big(b(b+1)\cdot k+b\big)+b}{b(b+1)} \\
&= \frac{b(b+1)\cdot(2kb+k)+2b^2+2b}{b(b+1)} \\
&= (2b+1)k+2.
\end{aligned}$$

Hence, $n$ is of the form $n = (2b+1)k+2$. This yields that $n \geqslant 2$ (because $k \geqslant 0$) and that $n-2$ is divisible by an odd integer greater than 1 (namely $2b+1 \geqslant 3$).
For the converse, suppose that for a number $n$ we have that $n \geqslant 2$ and that $n-2$ is divisible by an odd integer greater than 1, say $2b+1$ where $b \geqslant 1$ is an integer. Then there is a $k \geqslant 0$ for which we have $n = (2b+1)k+2$.

Now choose $a = b(b+1)\cdot k + b$, then $a$ is a positive integer and it holds that

$$\frac{a}{b} + \frac{a+1}{b+1} = \big((b+1)k+1\big) + \big(bk+1\big) = (2b+1)k + 2 = n.$$

We conclude that the numbers that can be written as $\frac{a}{b} + \frac{a+1}{b+1}$, where $a$ and $b$ are positive integers, are exactly the numbers $n \geqslant 2$ for which $n-2$ is divisible by an odd integer greater than 1. Hence, the numbers that cannot be written like this, are exactly 1 and the numbers $n \geqslant 2$ for which $n-2$ has no odd divisor greater than 1, otherwise stated for which $n-2$ is a power of two, say $2^m$ where $m \geqslant 0$.
We conclude that the numbers that cannot be written as $\frac{a}{b} + \frac{a+1}{b+1}$, are exactly 1 and the numbers of the form $2^m + 2$ for $m \geqslant 0$. □

**4.** Let $t \in \{1, 2, \ldots, n\}$. Suppose that $a_t = t$, then we may choose $i = j = t$ to find that $a_t + t = n+1$, hence $2t = n+1$. However, $n$ is divisible by 4, hence $n+1$ is odd. This is a contradiction. Now suppose that $a_t = n+1-t$. Then we may choose $i = n+1-t$ and $j = t$ to find that $a_{n+1-t} + t = n+1$, hence $a_{n+1-t} = n+1-t$. We have just seen that this cannot occur.
Now suppose that $a_t = u$ where $u \neq t$, $u \neq n+1-t$. Then we may choose $i = u$ and $j = t$ to find that $a_u + t = n+1$, hence $a_u = n+1-t$. Next, we may choose $i = n+1-t$ and $j = u$ to find that $a_{n+1-t} = n+1-u$. Next, we may choose $i = n+1-u$ and $j = n+1-t$ to find that $a_{n+1-u} = n+1-(n+1-y) = y$. Altogether, we have found that:

$$\begin{aligned} a_t &= u, \\ a_u &= n+1-t, \\ a_{n+1-t} &= n+1-u, \\ a_{n+1-u} &= t. \end{aligned}$$

Since $u \neq t$ and $u \neq n+1-t$, the four numbers on the right hand side are pairwise distinct. Furthermore, we can divide the four numbers into two pairs of the form $(v, n+1-v)$. We have found four numbers for which it holds that on the same positions in the permutation are the same four numbers, but in a different order. Now we may choose a $t'$ not equal to one of the four mentioned numbers and a $u'$ with $a_{t'} = u'$ and in the same way we find a quadruple containing $t'$. Note that $n+1-t'$ and $n+1-u'$ cannot be a number of the first quadruple, since then also $u'$ and $t'$ would be a number of the first quadruple. We can continue this way until all $n$ numbers have been divided into quadruples.
We can generate all permutations by using the following procedure:

- Choose the smallest number $k$ for which $a_k$ is not yet determined. Choose $a_k = u$ for a certain $u$ for which $a_u$ is not yet determined and for which $u \neq k$, $u \neq n+1-k$. This also determines the values of $a_u$, $a_{n+1-u}$ and $a_{n+1-k}$.
- Repeat the last step until all values of $a_k$ are determined.

For the first $k$ we have $n-2$ possible choices of $u$. During the next step we have $n-6$ possible choices left. One step later, we have $n-10$, and so on. Hence, the total number of permutations having the property, is equal to

$$2 \cdot 6 \cdot 10 \cdot \ldots \cdot (n-10) \cdot (n-6) \cdot (n-2).$$

Write $n = 4m$, then we can write this as

$$2^m \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m-5) \cdot (2m-3) \cdot (2m-1) = 2^m \cdot \frac{(2m)!}{2 \cdot 4 \cdot \ldots \cdot (2m)}$$

$$= \frac{(2m)!}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot m} = \frac{(2m)!}{m!} = \frac{(\frac{1}{2}n)!}{(\frac{1}{4}n)!}. \qquad \square$$

**5.** The fact that the angle bisector of angle $ABC$ intersects the short arc $AC$ in $P$, implies that $P$ lies exactly on the middle of this arc $AC$. We have to prove that $KL$ also goes through $P$, hence that $KL$ intersects the arc $AC$ in the middle. Because $K$ lies on $\Gamma$, this means that we have to prove that $KL$ is the angle bisector of $\angle AKC$.
Let $S$ be the intersection point of $B_1K$ and $BC$ and let $T$ be the intersection point of $BL$ and $AK$. Then we have $\angle BSK = 90°$ and $\angle BTK = 90°$, hence $BTSK$ is a cyclic quadrilateral. This yields that

$$\angle CBL = \angle SBT = \angle SKT = \angle B_1KA. \tag{2}$$

Because $ABKC$ is a cyclic quadrilateral, we have $\angle B_1AK = \angle CAK = \angle CBK$. By the exterior angle theorem, we have $\angle LB_1K = \angle B_1AK + \angle B_1KA$, hence because of (2) we get

$$\angle LB_1K = \angle B_1AK + \angle B_1KA = \angle CBK + \angle CBL = \angle LBK.$$

This yields that $LKBB_1$ is a cyclic quadrilateral, which means that $\angle LBB_1 = \angle LKB_1$. If we add (2) to this, we get

$$\angle CBB_1 = \angle CBL + \angle LBB_1 = \angle B_1KA + \angle LKB_1 = \angle LKA.$$

Hence,

$$\angle LKA = \angle CBB_1 = \tfrac{1}{2}\angle CBA = \tfrac{1}{2}\angle CKA,$$

where we used that $ABKC$ is a cyclic quadrilateral. This yields that $KL$ is the angle bisector of $\angle AKC$, as desired. $\square$

# IMO Team Selection Test 2, June 2012

## Problems

**1.** For all positive integers $a$ and $b$, we define $a \ominus b = \dfrac{a-b}{\gcd(a,b)}$.
Show that for every integer $n > 1$, the following holds: $n$ is a prime power if and only if for all positive integers $m$ such that $m < n$, it holds that $\gcd(n, n \ominus m) = 1$.

**2.** There are two boxes containing balls. One of them contains $m$ balls, and the other contains $n$ balls, where $m, n > 0$. Two actions are permitted:

(i) Remove an equal number of balls from both boxes.

(ii) Increase the number of balls in one of the boxes by a factor $k$.

Is it possible to remove all of the balls from both boxes with just these two actions,

1. if $k = 2$?
2. if $k = 3$?

**3.** Determine all pairs $(x, y)$ of positive integers satisfying

$$x + y + 1 \mid 2xy \qquad \text{and} \qquad x + y - 1 \mid x^2 + y^2 - 1.$$

**4.** Let $\triangle ABC$ be a triangle. The angle bisector of $\angle CAB$ intersects $BC$ at $L$. On the interior of line segments $AC$ and $AB$, two points, $M$ and $N$, respectively, are chosen in such a way that the lines $AL$, $BM$ and $CN$ are concurrent, and such that $\angle AMN = \angle ALB$. Prove that $\angle NML = 90°$.

**5.** Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f\big(x + xy + f(y)\big) = \big(f(x) + \tfrac{1}{2}\big)\big(f(y) + \tfrac{1}{2}\big)$$

for all $x, y \in \mathbb{R}$.

## Solutions

**1.** First suppose that $n = p^k$ with $p$ prime and $k > 0$. Then we need to show that for all positive integers $m$ with $m < n$, we have $\gcd(n, n \ominus m) = 1$. So let $m$ be a positive integer such that $m < n$. Write $m = p^l s$ with $p \nmid s$ and $0 \leqslant l < k$. Then $\gcd(n, m) = \gcd(p^k, p^l s) = p^l$, so

$$n \ominus m = \frac{p^k - p^l s}{p^l} = p^{k-l} - s.$$

Since $k - l \geqslant 1$, we have $\gcd(p^{k-l} - s, p) = \gcd(-s, p) = 1$, hence $\gcd(n, n \ominus m) = \gcd(p^k, p^{k-l} - s) = 1$.

Now suppose that $n$ is not a prime power. Then we need to show that there exists some positive integer $m$ with $m < n$ such that $\gcd(n, n \ominus m) \neq 1$. Let $q$ be the smallest prime dividing $n$, and let $t$ be the unique positive integer such that $q^t \mid n$ and $q^{t+1} \nmid n$. As $n$ is not a power of $q$, there exists a prime $p$ with $p > q$ dividing $n$. Hence $\frac{n}{q^t} \geqslant p > q$, from which follows that $n > q^{t+1}$. Now let $m = n - q^{t+1}$. Then we have $\gcd(n, m) = \gcd(n, q^{t+1}) = q^t$, so

$$n \ominus m = \frac{n - (n - q^{t+1})}{q^t} = \frac{q^{t+1}}{q^t} = q.$$

As $q \mid n$, it follows that $\gcd(n, n \ominus m) = q > 1$. □

**2.** First suppose that $k = 2$. Then we can remove all of the balls from both boxes as follows.

If $m = n$, we remove $m$ balls from both boxes, and we are done. If $m \neq n$, we may assume without loss of generality that $m < n$. If furthermore, $2m < n$ holds, then we double the number of balls in the first box, until there are $m'$ balls in the first box, where $m'$ satisfies both $m' < n$ and $2m' \geqslant n$. Hence we may assume without loss of generality that $2m \geqslant n$. Now remove $2n - m \geqslant 0$ balls from both boxes. In the first box, $m - (2m - n) = n - m > 0$ balls remain; in the second box, $n - (2m - n) = 2(n - m) > 0$ balls remain. Doubling the number of balls in the first box, then removing $2(n - m)$ balls from both boxes now makes both boxes empty.

Now consider the case $k = 3$. Let $S$ denote the number of balls in the first box, minus the number of balls in the second one. The first action does not change $S$. Applying the second action to some box with $t$ balls, does not change the parity of $S$ either, since we have $3t \equiv t \pmod 2$. Hence the parity of $S$ does not change. We conclude that the parity of $S$ is an invariant under both actions. Now suppose that we start with

$(m,n)=(1,2)$, then $S=-1\equiv 1 \pmod 2$. Then it is impossible to reach the situation in which both boxes are empty, since then one would have $S\equiv 0 \pmod 2$. Hence we cannot empty both boxes if $k=3$. $\square$

**3.** Note that

$$(x^2+y^2-1)-(x+y+1)(x+y-1)=(x^2+y^2-1)-(x^2+y^2+2xy-1)=-2xy.$$

As $x+y+1$ divides $x^2+y^2-1$ and $(x+y+1)(x+y-1)$, it follows that $x+y-1$ divides $2xy$. It was given that $x+y+1$ divides $2xy$ as well. But the difference of $x+y-1$ and $x+y+1$ is 2, so their greatest common divisor is either 1 or 2. So either $2xy$ is divisible by $(x+y+1)(x+y-1)$ (if the gcd is 1), or $2xy$ is divisible by $\frac{1}{2}(x+y+1)(x+y-1)$ (if the gcd is 2). In both cases, we have $(x+y-1)(x+y+1)\mid 4xy$, so for some $k\geqslant 1$, we have

$$4xy=k(x+y-1)(x+y+1)=k(x^2+y^2+2xy-1)\geqslant k(4xy-1).$$

Note that the last inequality holds, as for all real numbers $x$ and $y$, we have $x^2+y^2\geqslant 2xy$.

If $k\geqslant 2$, then $4xy\geqslant 2\cdot(4xy-1)$, so $4xy\leqslant 2$. This contradicts $x$ and $y$ being positive integers. Hence $k=1$, so we deduce that $4xy=x^2+y^2+2xy-1$. This implies that $x^2+y^2-1-2xy=0$, or equivalently, $(x-y)^2=1$. Hence either $x=y-1$, or $x=y+1$.

Hence the only pairs that can satisfy the conditions are $(x,x+1)$ with $x\geqslant 1$, and $(x,x-1)$ with $x\geqslant 2$. We check that they indeed satisfy the conditions. Indeed, for the first family of pairs, we note that $2x+2$ divides $2x(x+1)$, and that $2x$ divides $x^2+(x+1)^2-1=2x^2+2x$, and similarly for the second family of pairs. We conclude that the solutions are precisely the pairs $(x,x+1)$ with $x\geqslant 1$ and the pairs $(x,x-1)$ with $x\geqslant 2$. $\square$

**4.** Let $T$ be the intersection of $MN$ and $BC$. As $\angle ACB=\angle ALB-\angle LAC=\angle AMN-\angle LAC<\angle AMN$, it follows that $T$ and $B$ lie on the same side of $C$, and that $T$ and $N$ lie on the same side of $N$. Since $\angle AMT=\angle AMN=\angle ALB=\angle ALT$, we note that $AMLT$ is a cyclic quadrilateral. Hence $\angle NML=\angle TML=\angle TAL$. So it suffices to show that $\angle TAL=90^\circ$. This is the case if and only if $AT$ is the external angle bisector of $\angle CAB$, because the internal and external angle bisectors of any angle are perpendicular.

As $AL$, $BM$ and $CN$ are concurrent, Ceva's Theorem gives

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=1.$$

As $M$, $N$ and $T$ are collinear, Menelaus' Theorem gives

$$\frac{BT}{TC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1.$$

From these two identities, it follows that

$$\frac{BL}{LC} = -\frac{BT}{TC}.$$

By the angle bisector theorem, we have $\frac{|BL|}{|LC|} = \frac{|BA|}{|CA|}$, hence $\frac{|BT|}{|TC|} = \frac{|BA|}{|CA|}$. So, again by the angle bisector theorem, it follows that $AT$ is the external angle bisector of $\angle CAB$, as desired. $\square$

**5.** Substituting $y = -1$ gives

$$f\big(f(-1)\big) = \big(f(x) + \tfrac{1}{2}\big)\big(f(-1) + \tfrac{1}{2}\big).$$

If $f(-1) \neq -\frac{1}{2}$, then we can divide this equation by $f(-1)+\frac{1}{2}$, which gives

$$f(x) + \tfrac{1}{2} = \frac{f\big(f(-1)\big)}{f(-1) + \frac{1}{2}},$$

so $f$ is constant. Then there is some $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in \mathbb{R}$. Substituting this in the functional equation, gives

$$c = \big(c + \tfrac{1}{2}\big)\big(c + \tfrac{1}{2}\big),$$

or equivalently, $0 = c^2 + \frac{1}{4}$, but this equation has no real solutions. We deduce that $f$ cannot be constant.

Hence it follows that $f(-1) = -\frac{1}{2}$. This also implies that $f\big(f(-1)\big) = 0$, so $f(-\frac{1}{2}) = 0$. Substituting $x = 0$ and $y = -\frac{1}{2}$ then gives

$$f\big(f(-\tfrac{1}{2})\big) = \big(f(0) + \tfrac{1}{2}\big)\big(f(-\tfrac{1}{2}) + \tfrac{1}{2}\big),$$

hence

$$f(0) = \big(f(0) + \tfrac{1}{2}\big) \cdot \tfrac{1}{2},$$

so $f(0) = \frac{1}{2}$.

Now suppose that there exists some $a \neq -1$ such that $f(a) = -\frac{1}{2}$. Substituting $y = a$ now gives

$$f\big(x(1 + a) - \tfrac{1}{2}\big) = 0.$$

As $1+a \neq 0$, the function mapping $x$ to $x(1+a)-\frac{1}{2}$ attains all values in $\mathbb{R}$. Hence $f$ must be identically zero, hence constant, contradicting our earlier findings. So the unique $x \in \mathbb{R}$ such that $f(x) = -\frac{1}{2}$ is $x = -1$.

Now let $b$ be such that $f(b) = 0$. Substitute $x = b - \frac{1}{2}$, and $y = 0$:

$$f\big(b - \tfrac{1}{2} + \tfrac{1}{2}\big) = \big(f\big(b - \tfrac{1}{2}\big) + \tfrac{1}{2}\big)\big(\tfrac{1}{2} + \tfrac{1}{2}\big),$$

or equivalently,

$$f(b) = \big(f\big(b - \tfrac{1}{2}\big) + \tfrac{1}{2}\big).$$

Since $f(b) = 0$, we have $f(b - \frac{1}{2}) = -\frac{1}{2}$. Hence it follows by an earlier result that $b - \frac{1}{2} = -1$, hence $b = -\frac{1}{2}$. Hence the unique $x \in \mathbb{R}$ with $f(x) = 0$ is $x = -\frac{1}{2}$.

Now substitute $x = -1$. This gives

$$f\big(-1 - y + f(y)\big) = 0.$$

From this it follows that $-1 - y + f(y) = -\frac{1}{2}$, so $f(y) = y + \frac{1}{2}$. Hence if a function $f$ satisfies the functional equation, it must be the function given by $f(x) = x + \frac{1}{2}$.

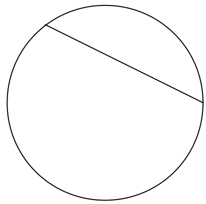
We check that this function indeed satisfies the functional equation. Indeed, the left hand side becomes $x + xy + y + 1$, which is equal to the right hand side $(x+1)(y+1)$. Hence the function $f$ given by $f(x) = x + \frac{1}{2}$ is the unique function satisfying the functional equation. $\square$

# Junior Mathematical Olympiad, October 2011

## Problems

### Part 1

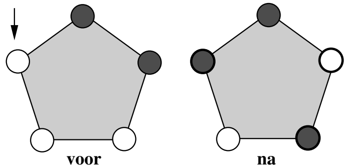
**1.** A *chord* of a circle is a line segment whose endpoints lie on the circle.



What is the maximal number of intersections that six chords of a circle can have?

A) 10 B) 12 C) 13 D) 14 E) 15

**2.** There are five buttons on the vertices of the Pentagon. Each of these are either black or white. Pressing one of these buttons will cause this button, together with the buttons on the edge opposite to it, to change colours; from black to white, and vice versa.




Suppose that all of the buttons are white. What is the minimal number of button presses needed to turn all of the buttons black?

A) 3 B) 4 C) 5 D) 7 E) 10

**3.** Let $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$ be a sequence of numbers with the property that any three consecutive numbers sum to 30 (e.g. $b + c + d = 30$). Suppose that $c = 5$.

What is $a + h$?

A) 10 B) 14 C) 15 D) 20 E) 25

**4.** In a cube with edges of length 3 cm, some holes are made. In the center of each face, a square hole with sides of length 2 cm is made, such that the sides of this square are parallel to the edges of the cube. This hole runs through the entire cube.

What is (in $cm^3$) the volume of the object that remains after making these three holes?

A) 6 B) 7 C) 8 D) 9 E) 10

**5.** In a classroom, one or more pupils *always* speak the truth. The other pupils sometimes do speak the truth, and sometimes do not. The pupils were asked how many of them always speak the truth. The answers were: 5, 6, 2, 3, 4, 6, 3, 6, 3, 4, 6, 5, 4, 3, and 6.

How many pupils do always speak the truth?

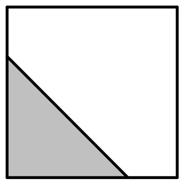
A) 2 B) 3 C) 4 D) 5 E) 6

**6.** Keith owns a machine in which he can put two numbers, $A$ and $B$. He can choose $A$ and $B$ to be any of the numbers $0, 1, 2, \ldots, 1000$ ($A$ and $B$ are allowed to be equal). The machine will then return the number $100 \times A + 3 \times B$. Keith tries to get the machine to return as many of the numbers from 1 up to 1000 as possible. However, for some numbers, it is impossible for the machine to return it, no matter what numbers Keith puts into the machine.

How many such numbers from 1 up to 1000 are there?

A) 66 B) 67 C) 99 D) 100 E) 363

**7.** A square with sides of length 1 is divided into two pieces by a line segment of length 1, parallel to a diagonal.



What is the area of the smallest piece?

A) $\frac{1}{4}$ B) $\frac{2}{7}$ C) $\frac{5}{16}$ D) $\frac{1}{3}$ E) $\frac{3}{8}$

**8.** In the addition to the right, the question mark and the asterisks all represent a digit. Every digit from 1 up to 9 occurs exactly once.

$$\begin{array}{cccc} * & * & * & \\ * & * & * & + \\ \hline 4 & ? & 9 & \end{array}$$

What is the digit represented by the question mark?

A) 2 B) 3 C) 5 D) 7 E) 8

**9.** Adrian puts stones in the nine squares of a $3 \times 3$-board. Into each of the squares, he is allowed to put any number of stones. He is also allowed to leave squares empty. When Adrian is finished, he counts the numbers of stones lying in each of the columns and rows. He wants these six numbers to be pairwise distinct.

At least how many stones does Adrian need to achieve this?

A) 7 B) 8 C) 9 D) 10 E) 11

**10.** Peter received 100 euro for his birthday. He uses all of it to buy exactly 100 objects. What he buys, consists of liquorice wheels of 10 cents a piece, bouncing balls of 2 euro a piece, and decks of playing cards of 5 euro a piece.

How many decks of playing cards does Peter buy?

A) 10 B) 11 C) 15 D) 16 E) 19

**11.** How many integers have the following properties; the digits are pairwise distinct, they are non-zero, and the sum of the digits is 10?

A) 32 B) 48 C) 56 D) 511 E) 512

**12.** Eight children together enumerate all of the numbers from 1 up to 2011, in the following manner.

- Angie enumerates all of the numbers from 1 up to 2011 in groups of three, skipping the middle number of each triple. So she says: "$1, 3, 4, 6, 7, 9, \ldots, 2005, 2007, 2008, 2010, 2011$".
- Ben enumerates all of the numbers Angie skipped, in groups of three, skipping the middle number of each triple.
- Catherine enumerates all of the numbers that were skipped by both Angie and Ben, again in groups of three, and skipping the middle number of each triple.
- Dorothy, Eve, Francis, and Gerald continue in the same way.
- In the end, Henry enumerates the only remaining number.

Which number is enumerated by Henry?

A) 712 B) 1094 C) 1123 D) 1265 E) 1387

**13.** In this magic square, the three rows, the three columns, and the two diagonals all have the same sum.

Which number is represented by the question mark?

A) 5 B) 6 C) 7 D) 10 E) 16

| | | |
|---|---|---|
| | 2 | |
| ? | | |
| | | 9 |

**14.** Ten ants walk upwards on a long and thin blade of grass. When an ant reaches the tip, it turns around. The ants cannot pass each other, so when they 'collide', they both turn around as well. In the end, all ten ants reach the bottom of the blade of grass safely.

What is the total number of times that an ant turned around?

A) 45 B) 50 C) 90 D) 95 E) 100

**15.** On a large sheet of squared paper, the natural numbers $1, 2, 3, 4, \ldots$ are written along a spiral, as shown in the figure. Somewhere on this sheet, the number 1000 is written with its eight neighbours around it.

| | | | | |
|---|---|---|---|---|
| 17 | 16 | 15 | 14 | 13 |
| 18 | 5 | 4 | 3 | 12 |
| 19 | 6 | 1 | 2 | 11 |
| 20 | 7 | 8 | 9 | 10 |
| 21 | 22 | ... | ... | ... |

Among these eight neighbouring numbers, which one is the smallest?

A) 873 B) 874 C) 875 D) 876 E) 877

## Part 2

The answer to each problem is a number.

**1.** Use the digits 4, 5, 6, and 7 to make two numbers (every digit may only be used once in total), such that the first number is a multiple of the second one.

What is the largest possible ratio between these two numbers?

**2.** The three sides of a triangle have three consecutive integers as length. The length of the shortest side is 30% of that of the circumference of the triangle.

What is the length of the longest side?

**3.** A group of 10 boys and 9 girls goes to watch a movie. They all buy a ticket for 6 euro. Fourteen of these 19 children buy a bag of popcorn. After the movie finished, it turns out that the combined amount of money that the boys spent (on the tickets and bags of popcorn), is equal to the combined amount of money that the girls spent.

Determine all of the possible prices of a bag of popcorn.

**4.** Quintijn writes down four pairwise distinct numbers that sum up to 44. Of every pair of numbers, he calculates their difference. These are: 1, 3, 4, 5, 6 and 9.

What possibilities are there for the value of the largest number of the four numbers that were written down? Write down all possibilities.

**5.** Julian has a big jar of liquorice wheels. Every day, he eats exactly a quarter of the liquorice wheels that are in the jar at the beginning of the day. At the end of the third day, less than 100 liquorice wheels remain in the jar.

At most how many liquorice wheels were in the jar at the beginning of the first day?

**6.** The number 1782379 has seven digits. Subtract this number from 9999999, and you get another seven digit number. Put this number at the end of 1782379, giving a fourteen digit number. Divide this number by 9999999.

What is the result?

**7.** At the beginning of class, all boys are present, but only two girls are present. After a while, three more girls enter class. This doubles the percentage of girls in class.

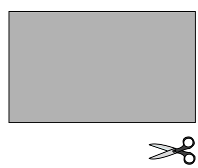
How many boys are in class?

**8.** We denote by $\max(a, b)$ the largest of the numbers $a$ and $b$. E.g. $\max(4, -7) = 4$.

What is the smallest possible value of $\max\big(5 - \max(a, 3), a + 3\big)$?

**9.** Divide a 3-by-5 rectangle with one straight cut in such a way, that a rhombus can be formed with the two pieces. A *rhombus* is a quadrilateral with four sides of equal length.

What is the length of the cut?



**10.** Given a three digit number, we do the following. We multiply all of its digits, and subtract the three digits one by one from the result.

Find the smallest three digit number such that this procedure gives the number 221.

# Solutions

## Part 1

**1.** E) $15$

**2.** C) $5$

**3.** E) $25$

**4.** B) $7$

**5.** B) $3$

**6.** C) $99$

**7.** A) $\frac{1}{4}$

**8.** C) $5$

**9.** B) $8$

**10.** B) $11$

**11.** C) $56$

**12.** B) $1094$

**13.** E) $16$

**14.** E) $100$

**15.** D) $876$

## Part 2

**1.** 189

**2.** 11

**3.** 1.50 and 3 euro

**4.** 15 and 16

**5.** 192

**6.** 1782380

**7.** 10

**8.** 2

**9.** 5

**10.** 568

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