$\begin{aligned} & 0(x)=\frac{-1}{x^{2}}--3 \\ & 0(x)=2 r^{-3}\end{aligned}$
$4(y)-6 x^{-9} f^{-2}(0)-6$
$\int^{2}(y)--6=-2 f^{2}(1)=24$
14. $\frac{1}{10}$
$\begin{aligned} & 2^{2}+\frac{x^{4}}{2^{1}} \\ & \therefore 1 \cdot \frac{6^{4}}{4}\end{aligned}$
$s^{2}-\frac{6}{4}$.
$\begin{aligned} & 11^{2} \\ & 6 \\ & 6 \\ & 43 \\ & 4\end{aligned}$
Preferably unsolved ones...

# $49^{\text {th }}$ Dutch Mathematical Olympiad 2010 

## NEDERLANDSE WISKUNDE OLYMPIADE

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## Introduction

In 2010 the Dutch Mathematical Olympiad consisted of three rounds. The first round was held on 29 January 2010 at the participating schools. The paper consisted of eight multiple choice questions and four open-answer questions, to be solved within 2 hours. In total 4150 students of 226 secondary schools participated in this first round.

In March we organised a new round at ten universities in the country. This round contained five open-answer questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2,5 hours. Those students from grade $5(4, \leqslant 3)$ that scored $14(12,10)$ points or more on the first round (out of a maximum of 36 points) were invited to this new second round.

From those 599 participants to the second round, the best students were invited for the final round. Those students from grade $5(4, \leqslant 3)$ that scored $28(18,14)$ points or more on the second round (out of a maximum of 40 points) were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited.

We organised training sessions at the ten universities for the 154 students who had been invited for the final round. Former Dutch IMO-participants were involved in the training sessions at each of the universities.

Out of those 154, in total 147 participated in the final round on 17 September 2010 at Eindhoven University of Technology. This final round contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 49th edition 2010.

The 25 most outstanding candidates of the Dutch Mathematical Olympiad 2010 were invited to an intensive seven-month training programme, consisting of weekly problem sets. Also, the students met twice for a three-day training camp, three times for a day at the university, and finally for a six-day training camp in the beginning of June.

On 18 March 2011 the first selection test was held. The best ten students participated in the third Benelux Mathematical Olympiad (BxMO), held in Mersch, Luxembourg.

In June, out of those 10 students and 1 reserve candidate, the team for the International Mathematical Olympiad 2011 was selected by two team selection tests on 8 and 11 June 2011. A seventh, young, promising student was selected to accompany the team to the IMO. The team had a training camp on Texel, one of the Dutch Frisian Islands, from 9 until 16 July, together with the team from New Zealand.

For younger students we organised the third Junior Mathematical Olympiad in October 2010 at the VU University Amsterdam. The students invited to participate in this event were the 30 best students of grade 1, grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with fifteen multiple choice questions and one with ten open-answer questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

The Dutch team for IMO 2011 Amsterdam consists of

- Ragnar Groot Koerkamp (16 y.o.)
- Jeroen Huijben ( 15 y.o., observer C at IMO 2010)
- Madelon de Kemp (18 y.o., bronze medal in IMO 2010)
- Daniël Kroes (17 y.o., honourable mention in IMO 2010)
- Merlijn Staps (16 y.o., bronze medal in IMO 2010)
- Jetze Zoethout (16 y.o.)

We bring as observer C the promising young student

- Jeroen Winkel (14 y.o.)

The team is coached by

- Johan Konter (team leader), Utrecht University
- Sietske Tacoma (deputy leader), Utrecht University

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

## First Round, January 2010

## Problems

## A-problems

A1. Consider figures consisting of three circles and two lines. What is the maximal number of intersection points in such figures?
A) 15
B) 16
C) 18
D) 19
E) 20

A2. A test consists of six questions worth successively 1 to 6 points. If your answer to a question is correct, the number of points that that question is worth will be added to your score. If your answer is incorrect, that number of points will be deducted from your score. So, if you only answered questions 1,3 and 4 correctly, your score will be $1-2+3+4-5-6=-5$. What is the number of possible scores you can get for this test?
A) 20
B) 22
C) 41
D) 43
E) 64

A3. A regular hexagon $A B C D E F$ has area 1. What is the area of the kite $A C D E$ ?
A) $\frac{2}{3}$
B) $\frac{1}{2} \sqrt{3}$
C) $\frac{5}{6}$
D) $\frac{3}{4}$
E) $\frac{1}{4} \sqrt{6}$


A4. Three players play a game with coins. Each round the player (or one of the players) who has the largest amount of coins will put one coin in a vase and then he will give each of the other players one coin. The vase is empty when the game starts and the three players possess respectively 13,14 and 15 coins. The game ends when one of the players has lost all his coins. How many coins will be in the vase when the game ends?
A) 36
B) 37
C) 38
D) 39
E) 40

A5. What is the last digit of $\left(\left(\left(\left(7^{6}\right)^{5}\right)^{4}\right)^{3}\right)^{2}$ ?
A) 1
B) 3
C) 5
D) 7
E) 9

A6. Calculate $\left((\sqrt{2}+1)^{7}+(\sqrt{2}-1)^{7}\right)^{2}-\left((\sqrt{2}+1)^{7}-(\sqrt{2}-1)^{7}\right)^{2}$.
A) 2
B) 4
C) $8 \sqrt{2}$
D) 128
E) 512

A7. An odometer indicates that an car has driven 2010 km . The odometer consists of six gears and there are no decimals; so the odometer displays 002010. However, each of the gears misses the digit 4 and will hop from 3 to 5 directly. What is the actual number of kilometres that the car has driven?
A) 1409
B) 1467
C) 1647
D) 1787
E) 1809

A8. Thirty people with different lengths are positioned in a rectangle of six rows each containing five persons. From each row we select the shortest person and from these six shortest persons we select the tallest; that is Piet. We also select the tallest person from each row and select the shortest from these six tallest persons; that is Jan. Then we put all thirty people in one line ordered by their length; the shortest person is standing on the left end and the tallest on the right end of the line. On which position can Jan not stand?
A) 21 positions left of Piet
D) 19 positions right of Piet
B) 19 positions left of Piet
E) 21 positions right of Piet
C) next to Piet

## B-problems

The answer to each B-problem is a number.
B1. Seven equally long matches are situated on a table as illustrated. How many degrees is the indicated angle?


B2. How many positive integers $a$ exist such that dividing 2216 by $a$ gives remainder 29?

B3. A figure consists of a square $A B C D$ and a semicircle with diameter $A D$ outside of the square. The square has side length 1 . What is the radius of the circumscribed circle of the figure?


B4. On a board containing 28 rows and 37 columns a number will be written in red in each of the squares in the following way: in the top row the numbers 1 to 37 will be written from left to right, in the second row the numbers 38 to 74 , etcetera.
In green a number will be written in each of the squares in the following way: in the leftmost column the numbers 1 to 28 will be written from top to bottom, in the column next to that column the numbers 29 to 56 will be written, etcetera.
In the square in the top left corner the number 1 is written in red and green. Add the red numbers of the squares in which the numbers written in red and green are the same. What is the sum of these numbers?

## Solutions

## A-problems

A1. D) 19 Any two circles intersect in at most 2 points, a circle and a line intersect in at most 2 points and two lines intersect in at most 1 point. Therefore, the number of intersections cannot be more than $6+$
 $6+6+1=19$. You can easily check that this number of intersections can be realized, see for example the figure on the right.

A2. B) 22 The possible scores are exactly the odd numbers $-21,-19, \ldots, 19,21$. To see that only odd numbers can occur, first consider the perfect score: 21 points. Now for every wrong answer, an even number must be subtracted from it, leaving an odd number.

Conversely, all odd numbers between -21 and 21 are possible scores. Indeed, if none or exactly one question is answered incorrectly, the resulting scores are $21,19,17,15,13,11$ and 9 . Answering question 6 and one of the first four questions incorrectly results in scores of $7,5,3$ and 1. Similarly, answering at most one question correctly, or answering question 6 and one of the first four questions correctly, result in scores of $-21,-19, \ldots,-1$.

A3. B) $\frac{2}{3} \quad$ Connect the centre $M$ to $A, C$ and $E$. Also connect $C$ and $E$. Then the hexagon is split into six equal triangles, four of which form the kite.


A4. B) 37 After three rounds, every player has lost 1 chip, and the pool has gained 3 . After $12 \cdot 3$ rounds, the players have 1,2 and 3 chips left respectively and 36 chips are in the pool. In the next and final round, one more chip is added to the pool.

A5. A) 1 If two numbers have a 1 as their last digit, then so does their product. Since the last digit of $7^{4}=2401$ equals 1 , the same holds for every power of $7^{4}$. In particular, the last digit of $\left(\left(\left(\left(7^{6}\right)^{5}\right)^{4}\right)^{3}\right)^{2}=$ $7^{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}=\left(7^{4}\right)^{180}$ is equal to 1 .

A6. B) $4 \quad$ Let $a=(\sqrt{2}+1)^{7}$ and $b=(\sqrt{2}-1)^{7}$. Then the given expression is equal to

$$
\begin{aligned}
(a+b)^{2}-(a-b)^{2} & =4 a b=4(\sqrt{2}+1)^{7}(\sqrt{2}-1)^{7} \\
& =4((\sqrt{2}+1)(\sqrt{2}-1))^{7}=4 \cdot 1^{7}=4
\end{aligned}
$$

A7. B) 1467 After 9 miles, the second digit from the right increases. After nine increments, so after 9.9 miles, the third digit from the right increases. After $9 \cdot 9 \cdot 9=729$ miles the fourth digit from the right increases. As the display shows 002010, the total distance travelled must be equal to $2 \cdot 729+1 \cdot 9=1467$ miles.

A8. E) 21 places to the right of Paul Among 6 selected people, Paul is tallest. Therefore at least 5 people are shorter than Paul. James is shortest among 6 selected people, so at least 5 people are taller than James. James can therefore not be 21 places to the right of Paul, since then at least $5+1+$ $20+1+5=32$ people would be standing in line.
The other positions are indeed possible. Here we only show this for answer C: James and Paul stand next to each other. Number the people in increasing height from 1 to 30 . Put numbers $1,2,3,4$ and 10 in one row and put one of the numbers $5,6,7,8$ en 9 in each of the other rows. The rest may be distributed freely over the remaining positions. The shortest
people in the six rows are numbers $1,5,6,7,8$ and 9 . Therefore Paul has number 9. James has number 10, since he is the tallest of his row and in every other row, there is someone with a number bigger than 10 .

## B-problems

B1. 100 The outer four matches form a parallelogram. The four angles at the bases of the two isosceles triangles are therefore all equal. This angle, denoted by $\alpha$, equals 180 degrees minus the angle we are looking for. The inner three matches form an equilateral triangle (angles equal to 60 de-
 grees). As the angles of any triangle add up to 180 degrees, we have that $(180-\alpha)+2(120-\alpha)=180$. It follows that $\alpha=80$ and therefore the angle in the question measures 100 degrees.

B2.
4 The statement that dividing 2216 by $a$ leaves a remainder of 29 , is the same as saying that $a$ divides $2216-29=2187$ and is larger than 29 (the remainder is always smaller than the divisor $a$ ). The divisors of $2187=3^{7}$ larger than 29 are $81,243,729$ and 2187. There are 4 choices for $a$.

B3.
$\frac{5}{6}$ Let $O$ be the centre of the circumscribed circle, $E$ the point where it touches the semicircle and let $M$ be the midpoint of $B C$. Applying the Pythagorean theorem gives $|O C|^{2}=|M C|^{2}+$ $|O M|^{2}$. Since $|O M|=|E M|-|O E|=\frac{3}{2}-|O C|$ this implies that $|O C|^{2}=\left(\frac{1}{2}\right)^{2}+\left(\frac{3}{2}-|O C|\right)^{2}$. Solving this equation, we obtain $|O C|=\frac{5}{6}$.


B4. 5185 Number the rows of the array from top to bottom using 0 to 27 and the columns from left to right using 0 to 36 . Consider the position in row $r$ and column $k$. The red number written equals $1+k+37 r$ and the green number equals $1+r+28 k$. These two numbers are equal exactly when $36 r=27 k$, that is, if and only if $4 r=3 k$. As solutions we obtain for $r$ the multiples of three: $0,3, \ldots, 27$ and for $k$ the matching multiples of four $0,4, \ldots, 36$. The coloured numbers in the corresponding ten positions are $1,1+115,1+2 \cdot 115, \ldots, 1+9 \cdot 115$. Adding these numbers we find the solution: $(1+(1+9 \cdot 115)) \cdot 5=5185$.

## Second Round, March 2010

## Problems

## B-problems

The answer to each B-problem is a number.

B1. Alice has got five real numbers $a<b<c<d<e$. She takes the sum of each pair of numbers and writes down the ten sums. The three smallest sums are 32,36 and 37 , while the two largest sums are 48 and 51 . Determine $e$.

B2. Let $A B$ be a diameter of a circle. Point $C$ is the point on segment $A B$ such that
$2 \cdot|A C|=|B C|$. The points $D$ and $E$ lie on the circle such that $C D$ is perpendicular to $A B$ and such that $D E$ is also a diameter of the circle. Write the areas of the triangles $A B D$ and $C D E$ as $O(A B D)$ and $O(C D E)$. Determine the value of $\frac{O(A B D)}{O(C D E)}$.

B3. A 24-hour digital clock displays the times from 00:00:00 till 23:59:59 during the day. You can add the digits of the time on every second of the day; this will give you an integer. For example, at 13:07:14 you will get $1+3+$ $0+7+1+4=16$. When you write down this sum for every possible state of the clock and then take the average of all these numbers, what will be the result?

B4. For the infinite sequence of numbers

$$
0,1,2,2,1,-1,-2,-1,1,3, \ldots
$$

the following rule holds. For each four consecutive numbers $\ldots, a, b, c, d, \ldots$ of the sequence the number $d$ is equal to $c$ minus the smallest of the two numbers $a$ and $b$. For example, the ninth number of the sequence is equal to $-1-(-2)=1$ and the tenth number is equal to $1-(-2)=3$. Calculate the 100th number of this sequence.

B5. Raymond has got five coins. On the heads side of each coin is the number 1. On the tails sides of the coins are the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and $\frac{1}{6}$ respectively. Because every coin has got either its heads side or its tails side facing up, there are 32 ways to put the five coins on the table. Raymond multiplies the five numbers facing up for each of these 32 situations and writes down all results.
If Raymond adds up these 32 numbers, what will be the final result?

## C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. Determine all positive integers $n$ consisting of four digits for which it holds that $n$ plus the sum of the digits of $n$ is equal to 2010 .

C2. Line segment $A B$ has length 10. A point $C$ lies on line segment $A B$ such that $|A C|=6$ and $|C B|=4$. Two points $X$ and $Y$ lie on the same side of the line $A B$, such that $|Y B|=|Y C|=3,|X A|=8$ and $|X C|=6$. Determine the length of line segment $X Y$.

## Solutions

## B-problems

B1. Out of the ten sums $32,36,37, \ldots, 48,51$ the largest one is $d+e$ and the second largest one is $c+e$. Therefore $d+e=51$ and $c+e=48$. Furthermore $a+b$ is the smallest sum and $a+c$ the second smallest, so $a+b=32$ and $a+c=36$.
The third smallest sum could be either $a+d$ or $b+c$. However, we know that

$$
a+d=(a+c)+(d+e)-(c+e)=36+51-48=39
$$

Apparently $a+d$ is not the third smallest sum, so $b+c=37$.
Combining the things we have found so far yields

$$
2 e=2(c+e)-(a+c)-(b+c)+(a+b)=2 \cdot 48-36-37+32=55
$$

Thus, the answer is $e=\frac{55}{2}$.

B2. Let $M$ be the midpoint of the circle. Triangles $C D M$ and $C E M$ have equal areas, because they have bases of the same length $|D M|=|E M|$ and the same height. This yields

$$
O(C D E)=2 \cdot O(C D M)
$$

Since $|A C|=\frac{1}{3}|A B|$ and $|A M|=\frac{1}{2}|A B|$ are true, $|C M|=|A M|-|A C|=\frac{1}{2}|A B|-$ $\frac{1}{3}|A B|=\frac{1}{6}|A B|$. Because triangles $A B D$

and $C D M$ have the same height with respect to the bases $A B$ and $C M$,

$$
O(A B D)=6 \cdot O(C D M) .
$$

Combining the two equalities yields $\frac{O(A B D)}{O(C D E)}=\frac{6}{2}=3$.

B3. Let $S$ be the number of possible states of the clock (for this problem it is not necessary to calculate $S$ ). If we would write down only the last digit for each state, then we would write down each of the digits $\frac{1}{10} S$ times. Thus, the total sum of these last digits is $\frac{1}{10} S(0+1+2+\ldots+9)=\frac{45}{10} S=\frac{9}{2} S$. If we would do the same for the second last digits, then we would only write down the digits 0 to 5 each exactly $\frac{1}{6} S$ times. Thus, the total sum of these digits is $\frac{1}{6} S(0+1+\ldots+5)=\frac{15}{6} S=\frac{5}{2} S$.
Analogously the sum of the digits in the middle two positions is equal to $\frac{9}{2} S+\frac{5}{2} S$.
We have to pay special attention to the first two digits, representing the hours of the time, because every digit does not appear evenly frequent. However, each of the digit pairs $00,01, \ldots, 23$ appears exactly $\frac{1}{24} S$ times. Thus, the total sum of the first to digits is

$$
\begin{aligned}
& \frac{1}{24} S((0+0)+(0+1)+(0+2)+\ldots+(2+3)) \\
= & \frac{1}{24} S(2(0+1+\ldots+9)+10 \cdot 1+4 \cdot 2+(0+1+2+3)) \\
= & \frac{114}{24} S=\frac{19}{4} S .
\end{aligned}
$$

The total sum of all digits of all possible states is $2 \cdot \frac{9}{2} S+2 \cdot \frac{5}{2} S+\frac{19}{4} S=\frac{75}{4} S$. The average sum is the total sum divided by $S$ and that is $\frac{75}{4}\left(=18 \frac{3}{4}\right)$.

B4. It is easy to calculate some more numbers of the sequence:

$$
\begin{array}{lllllllllllll}
0, & 1, & 2, & 2, & 1, & -1, & -2, & -1, & 1, & 3, & 4, & 3, & 0, \\
0, & 3, & 6, & 6, & 3, & -3, & \cdots
\end{array}
$$

We now see a clear pattern: after fifteen terms the sequence repeats itself, but all the terms are three times as large. By considering the following argument, we can verify that the sequence indeed does follow this pattern. Each number of the sequence is uniquely determined by its three predecessors. Assume that three consecutive numbers in the sequence $a, b, c$ will repeat fifteen positions further, multiplied by 3 , however:

$$
\ldots, a, b, c, d, \ldots, 3 a, 3 b, 3 c, \ldots
$$

The successor of $3 c$ is (by definition) equal to $3 c$ minus the smallest of $3 a$ and $3 b$. That is exactly three times as much as: $c$ minus the smallest of $a$ and $b$. So this is exactly three times $d$. If three consecutive numbers in the sequence repeat themselves after fifteen positions (with an extra factor 3), this also applies to the next number and therefore the number after the next number, etcetera.
Because of this pattern we see that the hundredth number is equal to the tenth number (3), but then multiplied $\frac{90}{15}=6$ times by 3 . This yields: $3 \cdot 3^{6}=2187$.

B5. The results that Raymond will get correspond exactly to the terms of the expansion of the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{6}\right)$. Therefore, the total sum is

$$
\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{6}\right)=\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{6}=\frac{7}{2} .
$$

## C-problems

C1. Suppose that $n$ is an integer consisting of four digits. We use the notation $n=1000 a+100 b+10 c+d$, where $a, b, c$ and $d$ are the digits of $n$. Therefore, $0 \leqslant a, b, c, d \leqslant 9$ and because a number does not start with a zero, $a>0$. We now want to know for which $n$ it is true that $n+(a+b+c+d)=$ $1001 a+101 b+11 c+2 d=2010$. Because $1001 \cdot 3$ is already larger than 2010, $a$ has to be smaller than 3 , so $a=1$ or $a=2$. We try both possibilities.

1. If $a=1$, then $101 b+11 c+2 d=2010-1001 \cdot 1=1009$. We will determine the possible values of $b$.

It is true that $0 \leqslant 11 c+2 d \leqslant 13 \cdot 9=117$. Because $101 b=$ $1009-(11 c+2 d)$, this yields $1009 \geqslant 101 b \geqslant 1009-117=892$. We are looking for multiples of 101 between 892 and 1009, because $101 b$ is a multiple of 101 . The only multiple of 101 between 892 and 1009 is 909 and therefore $b$ must be equal to 9 .
Now $11 c+2 d=1009-909=100$ has to be true. We will determine the possible values of $b$.
It is true that $0 \leqslant 2 d \leqslant 2 \cdot 9=18$. Because $11 c=100-2 d$, this also yields $100 \geqslant 11 c \geqslant 100-18=82$. Since $11 c=100-2 d$ is even, $11 c$ has to be equal to an even multiple of 11 between 82 and 100. The only even multiple of 11 between 82 and 100 is 88 . Therefore, $c=8$ and $d=6$ have to be true.
We find $n=1986$ as a possible solution.
2. If $a=2$, then $101 b+11 c+2 d=2010-1001 \cdot 2=8$.

Now we see that $b=c=0$ and $d=4$.
We find $n=2004$ as possible solution.
Finally, we check if the solutions 1986 and 2004 really suffice. Indeed, $1986+1+9+8+6=2010$ and $2004+2+0+0+4=2010$.
So, $n=1986$ and $n=2004$ are the only two solutions.

C2. Let $D$ be the midpoint of $B C$ and $M$ the midpoint of $A X$.
Since $|A M|=|M X|,|A C|=|C X|$ and $|M C|=|M C|$, the triangles $A M C$ and $X M C$ are congruent (SSS). This yields $\angle C M A=\angle C M X$. Because $\angle C M A+\angle C M X=180^{\circ}$, this yields $\angle C M A=\angle C M X=90^{\circ}$.
Analogously, the triangles $C D Y$ and
$B D Y$ are congruent. This yields $\angle C D Y=\angle B D Y=90^{\circ}$.
It is true that $\frac{|C D|}{|A M|}=\frac{|C Y|}{|A C|}=\frac{1}{2}$. Combined with $\angle A M C=\angle C D Y=90^{\circ}$ this yields that triangles $A M C$ and $C D Y$ are similar (ssr).
We now know that $\angle A C M=\angle C Y D$ and because of that $\angle A C M+$ $\angle D C Y=\angle C Y D+\angle D C Y=180^{\circ}-\angle C D Y=90^{\circ}$. This yields that $\angle M C Y=180^{\circ}-\angle A C M-\angle D C Y=90^{\circ}$.
Draw the line through $Y$ perpendicular to $A X$. Let $F$ be the intersection point of this line with $A X$. We see that $M C Y F$ is a rectangle.
The Pythagorean theorem yields $|M C|^{2}=|X C|^{2}-|M X|^{2}=6^{2}-4^{2}=20$. It is also true that $|X F|=|M X|-|M F|=|M X|-|C Y|=1$ and $|F Y|=|M C|$. Applying the Pythagorean theorem again yields $|X Y|^{2}=$ $|F Y|^{2}+|F X|^{2}=|M C|^{2}+1=21$ and $|X Y|=\sqrt{21}$.

## Final Round, September 2010

## Problems

For these problems not only the answer is important; you also have to describe the way you solved the problem.

1. Consider a triangle $A B C$ such that $\angle A=90^{\circ}, \angle C=$ $60^{\circ}$ and $|A C|=6$. Three circles with centers $A, B$ and $C$ are pairwise tangent in points on the three sides of the triangle.
Determine the area of the region enclosed by the three circles (the grey area in the figure).

2. A number is called polite if it can be written as $m+(m+1)+\cdots+(n-1)+n$, for certain positive integers $m<n$. For example: 18 is polite, since $18=$ $5+6+7$. A number is called a power of two if it can be written as $2^{\ell}$ for some integer $\ell \geqslant 0$.
(a) Show that no number is both polite and a power of two.
(b) Show that every positive integer is polite or a power of two.
3. Consider a triangle $X Y Z$ and a point $O$ in its interior. Three lines through $O$ are drawn, parallel to the respective sides of the triangle. The intersections with the sides of the triangle determine six line segments from $O$ to the sides of the triangle.
The lengths of these segments are integer num-
 bers $a, b, c, d, e$ and $f$ (see figure).
Prove that the product $a \cdot b \cdot c \cdot d \cdot e \cdot f$ is a perfect square.
4. (a) Determine all pairs $(x, y)$ of (real) numbers with $0<x<1$ en $0<$ $y<1$ for which $x+3 y$ and $3 x+y$ are both integer. An example is $(x, y)=\left(\frac{3}{8}, \frac{7}{8}\right)$, because $x+3 y=\frac{3}{8}+\frac{21}{8}=\frac{24}{8}=3$ en $3 x+y=\frac{9}{8}+\frac{7}{8}=$ $\frac{16}{8}=2$.
(b) Determine the integer $m \geqslant 2$ for which there are exactly 119 pairs $(x, y)$ with $0<x<1$ en $0<y<1$ such that $x+m y$ en $m x+y$ are integer.
Remark: if $u \neq v$, the pairs $(u, v)$ and $(v, u)$ are different.
5. Amber and Brian are playing a game using 2010 coins. Throughout the game, the coins are divided into a number of piles of at least 1 coin each. A move consists of choosing one or more piles and dividing each of them into two smaller piles. (So piles consisting of only 1 coin cannot be chosen.) Initially, there is only one pile containing all 2010 coins. Amber and Brian alternatingly take turns to make a move, starting with Amber. The winner is the one achieving the situation where all piles have only one coin.
Show that Amber can win the game, no matter which moves Brian makes.

## Solutions

1. We recognize triangle $A B C$ to be half an equilateral triangle. This implies that $|B C|=2|A C|=$ 12. The Pythagorean theorem yields: $|A B|=$ $\sqrt{|B C|^{2}-|A C|^{2}}=\sqrt{108}=6 \sqrt{3}$.
Denote the pairwise tangent points of the three circles by $D, E$ and $F$ (see figure) and the radii of the three circles by $r_{A}, r_{B}$ and $r_{C}$. The strategy will be to determine the area of the three circular sectors and subtract them from the area of triangle $A B C$.


We see that $2 r_{A}=\left(r_{A}+r_{C}\right)+\left(r_{A}+r_{B}\right)-\left(r_{B}+r_{C}\right)=|A C|+|A B|-|B C|=$ $6 \sqrt{3}-6$, so $r_{A}=3 \sqrt{3}-3$. It follows that $r_{B}=6 \sqrt{3}-r_{A}=3 \sqrt{3}+3$ en $r_{C}=6-r_{A}=9-3 \sqrt{3}$.
The area of a circle of radius $r$ equals $\pi r^{2}$. Therefore, the area of circular sector $A F E$ equals $\frac{90}{360} \cdot \pi r_{A}^{2}$, or $\frac{1}{4} \pi(36-18 \sqrt{3})=9 \pi-\frac{9}{2} \sqrt{3} \pi$. For the area of circular sectors $B D F$ and $C E D$ we find, respectively, $\frac{30}{360} \pi r_{B}^{2}=3 \pi+\frac{3}{2} \sqrt{3} \pi$ and $\frac{60}{360} \pi r_{C}^{2}=18 \pi-9 \sqrt{3} \pi$.
Since $A B C$ has an area of $\frac{1}{2} \cdot|A B| \cdot|A C|=18 \sqrt{3}$, we obtain a value of $18 \sqrt{3}-\left(9 \pi-\frac{9}{2} \sqrt{3} \pi\right)-\left(3 \pi+\frac{3}{2} \sqrt{3} \pi\right)-(18 \pi-9 \sqrt{3} \pi)=18 \sqrt{3}-30 \pi+12 \sqrt{3} \pi$ for the area of the grey region.
2. (a) Suppose that $k=m+(m+1)+\cdots+(n-1)+n$ is a polite number. The sum formula for arithmetic sequences gives $k=\frac{1}{2}(m+n)(n-m+1)$. As $m$ and $n$ are different positive numbers, $m+n \geqslant 3$ and $(n-m)+1 \geqslant$ 2 must hold.
Since $(m+n)+(n-m+1)=2 n+1$ is odd, one of the numbers $m+n$ and $n-m+1$ is odd. Hence $2 k=(m+n)(n-m+1)$ has
an odd divisor (greater than 1) and is therefore not a power of two. This implies that $k$ is not a power of two either.
We conclude that no number can be both polite and a power of two.
(b) Suppose that $k$ is a positive integer, not a power of two. We will show $k$ to be a polite number. Collecting all factors 2 , we can write $k=c \cdot 2^{d}$, where $c$ is odd and $d \geqslant 0$ is a nonnegative integer. The assumption that $k$ is not a power of two, means that $c>1$. We need to find $n>m$ such that $m+\cdots+n=\frac{1}{2}(m+n)(n-m+1)=c \cdot 2^{d}$, or $(m+n) \cdot(n-m+1)=c \cdot 2^{d+1}$. We can achieve this by choosing $m$ en $n$ in such a way that $m+n=c$ and $n-m+1=2^{d+1}$, or conversely: $m+n=2^{d+1}$ and $n-m+1=c$. To ensure that $m$ will be positive, we consider two cases.
For $c \geqslant 2^{d+1}$ we solve: $m+n=c, n-m+1=2^{d+1}$. This gives $m=\left(c-2^{d+1}+1\right) / 2$ and $n=\left(c+2^{d+1}-1\right) / 2$. Obviously, $n>m$ (since $2^{d+1} \geqslant 2$ ). Both $m$ and $n$ are integers (the numerators are even since $c$ is odd) and positive by the assumption $c \geqslant 2^{d+1}$.
For $c<2^{d+1}$ we solve: $m+n=2^{d+1}, n-m+1=c$. This gives $m=\left(2^{d+1}-c+1\right) / 2$ and $n=\left(2^{d+1}+c-1\right) / 2$. Clearly, $n>m$ holds (since $c>1$ ) and both $m$ and $n$ are positive integers.
3. Since $A O$ and $X Z$ are parallel, $\angle O A B=\angle Z X Y$ are corresponding angles. Similarly, since $B O$ and $Y Z$ are parallel, $\angle A B O=\angle X Y Z$ holds. We deduce that $\triangle O A B \sim \triangle Z X Y$ (equal angles). Hence there is a scaling factor $u$ such that $a=u|X Z|$ and $b=u|Y Z|$. Using similar arguments we find that $\triangle O C D \sim \triangle X Y Z$ and $\triangle O E F \sim \triangle Y Z X$. So there are scal-
 ing factors $v$ and $w$ such that $c=v|X Y|, d=v|X Z|, e=w|Y Z|$ and $f=w|X Y|$.
We now see that $a \cdot c \cdot e=u v w \cdot|X Y| \cdot|Y Z| \cdot|Z X|=b \cdot d \cdot f$. This implies that $a \cdot b \cdot c \cdot d \cdot e \cdot f=(a \cdot c \cdot e)^{2}$, which is a perfect square since $a, c$ and $e$ are integers.
4. (a) Suppose that $(x, y)$ is such a pair and consider the integers $a=x+3 y$ and $b=3 x+y$. From $0<x, y<1$ it follows that $0<a, b<4$, or: $1 \leqslant a, b \leqslant 3$.
Conversely, let $a$ and $b$ be integers such that $1 \leqslant a, b \leqslant 3$. There is a unique pair of numbers $(x, y)$ that satisfies $a=x+3 y$ en $b=3 x+y$. Indeed, combining the two equations, we get $3 b-a=3(3 x+y)-$
$(x+3 y)=8 x$ and $3 a-b=8 y$. In other words $x=(3 b-a) / 8$ and $y=(3 a-b) / 8$ (and these $x$ and $y$ do satisfy the two equations). If we substitute $1,2,3$ for $a$ and $b$, we obtain the following nine pairs $(x, y)$ :

$$
\left(\frac{2}{8}, \frac{2}{8}\right),\left(\frac{5}{8}, \frac{1}{8}\right),\left(\frac{8}{8}, \frac{0}{8}\right), \quad\left(\frac{1}{8}, \frac{5}{8}\right),\left(\frac{4}{8}, \frac{4}{8}\right),\left(\frac{7}{8}, \frac{3}{8}\right), \quad\left(\frac{0}{8}, \frac{8}{8}\right),\left(\frac{3}{8}, \frac{7}{8}\right),\left(\frac{6}{8}, \frac{6}{8}\right) .
$$

The condition $0<x, y<1$ rules out the two candidates $(x, y)=\left(\frac{8}{8}, \frac{0}{8}\right)$ and $(x, y)=\left(\frac{0}{8}, \frac{8}{8}\right)$. This leaves the 7 pairs we were asked to find.
(b) Suppose that $0<x, y<1$ holds and that $a=x+m y$ and $b=m x+y$ are integers. Then $1 \leqslant a, b \leqslant m$ holds.
Given integers $a$ and $b$ with $1 \leqslant a, b \leqslant m$, there is a unique pair $(x, y)$ for which $x+m y=a$ and $m x+y=b$ hold. Indeed, combining the two equalities gives : $m b-a=\left(m^{2}-1\right) x$ and $m a-b=\left(m^{2}-1\right) y$, or: $x=(m b-a) /\left(m^{2}-1\right)$ and $y=(m a-b) /\left(m^{2}-1\right)$. These $x$ and $y$ indeed satisfy the two equations.
For given $a$ and $b$, we determine whether the corresponding numbers $x$ and $y$ satisfy $0<x, y<1$. From $1 \leqslant a, b \leqslant m$ it follows that $x \geqslant(m \cdot 1-m) /\left(m^{2}-1\right)=0$ and $x \leqslant(m \cdot m-1) /\left(m^{2}-1\right)=1$. The cases $x=0$ and $x=1$ exactly correspond to $(a, b)=(m, 1)$ and $(a, b)=(1, m)$ respectively. Similarly, $0<y<1$ holds, unless $(a, b)=(1, m)$ or $(a, b)=(m, 1)$. Among the $m^{2}$ possible pairs $(a, b)$, there are exactly two for which $(x, y)$ is not a solution. In total there are $m^{2}-2$ solutions $(x, y)$.
From $m^{2}-2=119$, we see that $m=11$.
5. A strategy that guarantees a win for Amber is as follows. In Amber's turn, she splits every pile with an even number of coins (say $2 k$ ) in two piles with an odd number of coins: 1 coin and $2 k-1$ coin respectively. The piles having an odd number of coins, she leaves untouched. So in her first turn, she created one pile of 1 coin and one of 2009 coins.
When Brian gets to make a move, all piles will have an odd number of coins. He is therefore forced to split an odd pile, creating a new pile with an even number of coins. This implies that Amber, in the next turn, can continue her strategy, since there will be at least one even pile.
With each turn, the number of piles increases, so after at most 2009 turns, the game is over. Since Brian always creates an even pile, the game cannot end during his turn. Therefore, it will be Amber who wins the game.

## BxMO Team Selection Test, March 2011

## Problems

1. All positive integers are coloured either red or green, such that the following conditions are satisfied:

- There are equally many red as green integers.
- The sum of three (not necessarily distinct) red integers is red.
- The sum of three (not necessarily distinct) green integers is green.

Find all colourings that satisfy these conditions.
2. In an acute triangle $A B C$ the angle $\angle C$ is greater than $\angle A$. Let $E$ be such that $A E$ is a diameter of the circumscribed circle $\Gamma$ of $\triangle A B C$. Let $K$ be the intersection of $A C$ and the tangent line at $B$ to $\Gamma$. Let $L$ be the the orthogonal projection of $K$ on $A E$ and let $D$ be the intersection of $K L$ and $A B$.
Prove that $C E$ is the bisector of $\angle B C D$.
3. Find all triples $(x, y, z)$ of real numbers that satisfy

$$
x^{2}+y^{2}+z^{2}+1=x y+y z+z x+|x-2 y+z| .
$$

4. Let $n \geqslant 2$ be an integer. Let $a$ be the greatest positive integer such that $2^{a} \mid 5^{n}-3^{n}$. Let $b$ be the greatest positive integer such that $2^{b} \leqslant n$. Prove that $a \leqslant b+3$.
5. A trapezoid $A B C D$ is given with $B C \| A D$. Assume that the bisectors of the angles $B A D$ and $C D A$ intersect on the perpendicular bisector of the line segment $B C$.
Prove that $|A B|=|C D|$ or $|A B|+|C D|=|A D|$.

## Solutions

1. Suppose that we colour the integer $k$ red. We prove by induction to $n$ that $(2 n+1) k$ then also is coloured red for all $n \geqslant 0$. For $n=0$ this is trivial. Suppose that $(2 n-1) k$ is red for a certain $n$, then $k+k+(2 n-1) k=$ $(2 n+1) k$ is also red. This concludes the induction. Analogously if $k$ is coloured green, then also $(2 n+1) k$ is coloured green for all $n \geqslant 0$.

Without loss of generality we may assume that 1 is red. Then all odd numbers are red. Now suppose that there is also an even number $2 m$ that is red. Because both 1 and $2 m$ are red, we can easily show by induction that $2 m+2 n$ is red for all $n \geqslant 0$. Then only finitely many numbers are left that could be green, namely the even numbers smaller than $2 m$. However, if one of these numbers is green, then also all odd multiples of that number are green and these are infinitely many numbers, yielding a contradiction. If on the other hand none of these numbers is green, then we have a contradiction with the first condition. We conclude that no even number is red. Therefore all even numbers are green.
This colouring also satisfies all conditions: the sum of three odd numbers is always odd and the sum of three even numbers is always even, which shows that the second and third condition are satisfied. The first condition is also clearly satisfied.
Hence the only possible colourings are: all even numbers are coloured green and all odd numbers red, or all odd numbers green and all even numbers red.
2. The conditions in the problem fix the configuration. Because $\angle L A D=$ $\angle E A B=\angle E C B$ by the inscribed angle theorem on chord $E B$ of circle $\Gamma$ :

$$
\begin{aligned}
\angle B D K & =\angle A D L=180^{\circ}-\angle D L A-\angle L A D=90^{\circ}-\angle L A D \\
& =90^{\circ}-\angle E C B=\angle E C A-\angle E C B=\angle B C A=180^{\circ}-\angle B C K .
\end{aligned}
$$

This yields that $B D K C$ is a cyclic quadrilateral. Therefore $\angle B C D=$ $\angle B K D$. Now we will prove that $\angle B K D=2 \angle B C E$, from which the problem follows.
By the inscribed angle theorem, $\angle E B K=\angle B C E$, hence $\angle A B K=$ $\angle A B E+\angle E B K=90^{\circ}+\angle B C E$. Together with the sum of the angles of triangle $A B C$ this yields

$$
\angle A K B=180^{\circ}-\left(90^{\circ}+\angle B C E\right)-\angle B A K=90^{\circ}-\angle B C E-\angle B A K .
$$

The inscribed angle theorem gives $\angle B A K=\angle E A K+\angle B A E=\angle E A K+$ $\angle B C E$, therefore we find

$$
\angle A K B=90^{\circ}-2 \angle B C E-\angle E A K=90^{\circ}-2 \angle B C E-\angle L A K .
$$

From the sum of the angles of triangle $A K L$ follows $\angle A K L=90^{\circ}-\angle L A K$, hence

$$
\angle A K B=\angle A K L-2 \angle B C E .
$$

This yields

$$
\angle B K D=\angle A K L-\angle A K B=2 \angle B C E,
$$

what we wanted to prove.
3. We rewrite the given equation as

$$
\frac{1}{2} x^{2}-x y+\frac{1}{2} y^{2}+\frac{1}{2} y^{2}-y z+\frac{1}{2} z^{2}+\frac{1}{2} z^{2}-z x+\frac{1}{2} x^{2}+1=|x-2 y+z|,
$$

or, equivalently,

$$
\begin{equation*}
\frac{1}{2}(x-y)^{2}+\frac{1}{2}(y-z)^{2}+\frac{1}{2}(z-x)^{2}+1=|(x-y)+(z-y)| . \tag{1}
\end{equation*}
$$

Now substitute $a=x-y$ and $b=z-y$. Then $x-z=a-b$, thus we get

$$
\begin{equation*}
\frac{1}{2} a^{2}+\frac{1}{2} b^{2}+\frac{1}{2}(a-b)^{2}+1=|a+b| \tag{2}
\end{equation*}
$$

From $(a-b)^{2} \geqslant 0$, it follows that $a^{2}-2 a b+b^{2} \geqslant 0$, hence $2 a^{2}+2 b^{2} \geqslant$ $a^{2}+b^{2}+2 a b$, which means that $a^{2}+b^{2} \geqslant \frac{(a+b)^{2}}{2}$ with equality if and only if $a=b$. Furthermore, also $(a-b)^{2} \geqslant 0$. Hence

$$
|a+b|=\frac{1}{2} a^{2}+\frac{1}{2} b^{2}+\frac{1}{2}(a-b)^{2}+1 \geqslant \frac{(a+b)^{2}}{4}+1 .
$$

Now write $c=|a+b|$, then the expression becomes

$$
c \geqslant \frac{c^{2}}{4}+1
$$

We can rewrite this to $c^{2}-4 c+4 \leqslant 0$, or, equivalently $(c-2)^{2} \leqslant 0$. Because the left hand side is a square, equality must hold, so $c=2$. Furthermore, in our previous inequalities equality also has to hold, so $a=b$. Substituting this in (2) gives $a^{2}+1=2$, so $a= \pm 1$. Thus we find the triples $(y+1, y, y+1)$ and $(y-1, y, y-1)$ for arbitrary $y \in \mathbb{R}$. Substituting this in the equation (1) (that is equivalent to the original equation) shows that these triples are indeed solutions for all $y \in \mathbb{R}$. Hence, all solutions are given by $(y+1, y, y+1)$ and $(y-1, y, y-1)$ with $y \in \mathbb{R}$.
4. First we prove the statement for all odd numbers $n$. In this case modulo 4 we have $5^{n} \equiv 1^{n}=1$ and $3^{n} \equiv(-1)^{n} \equiv-1$, hence $5^{n}-3^{n} \equiv 2 \bmod 4$. Therefore, if $n$ is odd, then $a=1$. Because $b \geqslant 1$, the inequality $a \leqslant b+3$ is satisfied.

Now suppose that $n \equiv 2 \bmod 4$. Write $n=2 k$ with $k$ an odd positive integer. Notice that $5^{2 k}-3^{2 k}=\left(5^{k}-3^{k}\right)\left(5^{k}+3^{k}\right)$. We just showed that $5^{k}-3^{k}$ has precisely one factor 2 , since $k$ is odd. Now consider $5^{k}+3^{k}$ modulo 16. For $m=1,2,3,4$ we have that $5^{m}$ modulo 16 is congruent to respectively $5,9,13,1$. Because $5^{4} \equiv 1 \bmod 16$, we have $5^{k} \equiv 5$ for all $k \equiv 1 \bmod 4$ and $5^{k} \equiv 13$ for all $k \equiv 3 \bmod 4$. For $m=1,2,3,4$ we have that $3^{m}$ modulo 16 is congruent to respectively $3,9,11,1$. Because $3^{4} \equiv 1 \bmod 16$, we have $3^{k} \equiv 3$ for all $k \equiv 1 \bmod 4$ and $3^{k} \equiv 11$ for all $k \equiv 3 \bmod 4$. Altogether $5^{k}+3^{k} \equiv 5+3 \equiv 8 \bmod 16$ if $k \equiv 1 \bmod 4$ and $5^{k}+3^{k} \equiv 13+11=24 \equiv 8 \bmod 16$ if $k \equiv 3 \bmod 4$. In both cases $5^{k}+3^{k}$ contains precisely 3 factors 2 .
We conclude that for $n \equiv 2 \bmod 4$ we have: $a=4$. Because $b \geqslant 1$, the inequality $a \leqslant b+3$ is now satisfied.
We will prove by induction to $m$ that $a \leqslant b+3$ for all positive numbers $n$ with precisely $m \geqslant 1$ factors 2 . The induction basis is $m=1$, that is the case $n \equiv 2 \bmod 4$, which we just solved.
Now let $m \geqslant 1$ and assume as induction hypothesis that we already showed that $a \leqslant b+3$ for all numbers $n$ with precisely $m$ factors 2 . Now consider a number $n$ with $m+1$ factors 2 . We write $n=2 k$, where $k$ has precisely $m$ factors 2 . For clarity let $a(k)$ and $b(k)$ be the $a$ and the $b$ corresponding to $k$, and $a(n)$ and $b(n)$ the $a$ and $b$ corresponding to $n$. The induction hypothesis yields $a(k) \leqslant b(k)+3$. Now we will prove that $a(n) \leqslant b(n)+3$. We have $5^{n}-3^{n}=5^{2 k}-3^{2 k}=\left(5^{k}-3^{k}\right)\left(5^{k}+3^{k}\right)$. Because $k$ is even (it contains $m \geqslant 1$ factors 2 ) modulo 4 we have $5^{k}+3^{k} \equiv 1^{k}+(-1)^{k} \equiv 2$ $\bmod 4$. Hence $5^{k}+3^{k}$ contains precisely one factor 2 . Furthermore, $5^{k}-3^{k}$ contains precisely $a(k)$ factors 2. Thus $a(n)=a(k)+1$. We already know that $2^{b(k)} \leqslant k$ and $2^{b(k)+1}>k$, which yields $2^{b(k)+1} \leqslant 2 k$ and $2^{b(k)+2}>2 k$. Hence $b(n)=b(k)+1$. Now we conclude: $a(n)=a(k)+1 \leqslant b(k)+3+1=$ $b(n)+3$, which concludes the induction.
This proves that $a \leqslant b+3$ for all integers $n \geqslant 2$.
5. Let $M$ be the midpoint of $B C$ and let $P$ be the intersection of the perpendicular bisector of $B C$ and $A D$. Let $K$ be the intersection of $M P$ and the two angle bisectors. Let $L$ and $N$ be the feet of the lines through $K$ perpendicular to sides $A B$ and $D C$, respectively. Because $A K$ and $D K$ are bisectors, $|K L|=|K P|=|K N|$. Furthermore $K$ also lies on the perpendicular bisector of $B C$, thus $|K B|=|K C|$. Because triangles $B L K$ and $C N K$ both have a right angle, they are congruent by (RHS).
Now we distinguish between four cases. First consider the case that $L$ and $N$ lie on the interior of respectively sides $A B$ and $D C$. Notice that triangle $K B C$ is isosceles, yielding $\angle K B C=\angle B C K$. Hence by $\triangle B L K \cong$ $\triangle C N K$ :

$$
\angle A B C=\angle L B K+\angle K B C=\angle N C K+\angle B C K=\angle B C D .
$$

From this follows that $A B C D$ is a isosceles trapezoid and hence $|A B|=$ $|C D|$.
In the case that $L$ and $N$ both lie on the exterior of sides $A B$ and $C D$ we can show analogously that $|A B|=|C D|$.
Now consider the case that $L$ lies on the interior of side $A B$, but $N$ on the exterior of side $D C$. Because $A K$ and $D K$ are bisectors, $|A L|=|A P|$ and $|D N|=|D P|$. Therefore by $\triangle B L K \cong \triangle C N K$ :

$$
\begin{aligned}
|A B|+|C D| & =(|A L|+|L B|)+(|D N|-|N C|) \\
& =|A P|+|L B|+|D P|-|L B|=|A D| .
\end{aligned}
$$

In the case that $L$ lies on the exterior of $A B$ and $N$ on the interior of $D C$, we analogously show that $|A B|+|C D|=|A D|$.

## Benelux Mathematical Olympiad, May 2011 Mersch, Luxembourg <br> Problems

1. An ordered pair of integers $(m, n)$ with $1<m<n$ is said to be a Benelux couple if the following two conditions hold: $m$ has the same prime divisors as $n$, and $m+1$ has the same prime divisors as $n+1$.
(a) Find three Benelux couples $(m, n)$ with $m \leqslant 14$.
(b) Prove that there exist infinitely many Benelux couples.
2. Let $A B C$ be a triangle with incentre $I$. The angle bisectors $A I, B I$ and $C I$ meet $[B C],[C A]$ and $[A B]$ at $D, E$ and $F$, respectively. The perpendicular bisector of $[A D]$ intersects the lines $B I$ and $C I$ at $M$ and $N$, respectively. Show that $A, I, M$ and $N$ lie on a circle.
3. If $k$ is an integer, let $c(k)$ denote the largest cube that is less than or equal to $k$. Find all positive integers $p$ for which the following sequence is bounded:

$$
a_{0}=p \quad \text { and } \quad a_{n+1}=3 a_{n}-2 c\left(a_{n}\right) \quad \text { for } n \geqslant 0 .
$$

(A sequence $a_{0}, a_{1}, \ldots$ of reals is said to be bounded if there exists an $M \in \mathbb{R}$ such that, for all $n \geqslant 0,\left|a_{n}\right| \leqslant M$.)
4. Abby and Brian play the following game: They first choose a positive integer $N$. Then they write numbers on a blackboard in turn. Abby starts by writing a 1 . Thereafter, when one of them has written the number $n$, the other writes down either $n+1$ or $2 n$, provided that the number is not greater than $N$. The player who writes $N$ on the blackboard wins.
(a) Determine which player has a winning strategy if $N=2011$.
(b) Find the number of positive integers $N \leqslant 2011$ for which Brian has a winning strategy.

## Solutions

1. (a) It is possible to see that $(2,8),(6,48)$ and $(14,224)$ are Benelux couples.
(b) Let $k \geqslant 2$ be an integer and $m=2^{k}-2$. Define $n=m(m+2)=$ $2^{k}\left(2^{k}-2\right)$. Since $m$ is even, $m$ and $n$ have the same prime factors. Also, $n+1=m(m+2)+1=(m+1)^{2}$, so $m+1$ and $n+1$ have the same prime factors as well. We have thus obtained a Benelux couple $\left(2^{k}-2,2^{k}\left(2^{k}-2\right)\right)$ for each $k \geqslant 2$.
2. The quadrilateral $A M D B$ is cyclic. Indeed, $M$ is the intersection of the line $B I$, which bisects the angle $\angle A B D$ in $A B D$ and the perpendicular bisector of $[A D]$. By uniqueness of this intersection point, it follows that $M$ lies on the circumcircle of $A B D$, and hence $A M D B$ is cyclic. Analogously, $A N D C$ is cyclic.
Since $A M D B$ and $A N D C$ are cyclic, $\angle A M I+\angle A N I=\angle A M B+\angle A N C=$ $\angle A D B+\angle A D C=180^{\circ}$, because $B$ and $M$, and $C$ and $N$ lie on either side of $A D$. Hence $A M I N$ is cyclic, for $M$ and $N$ lie on opposite sides of $A D$.
3. Since $c\left(a_{n}\right) \leqslant a_{n}$ for all $n \in \mathbb{N}, a_{n+1} \geqslant a_{n}$, where equality holds if and only if $c\left(a_{n}\right)=a_{n}$. Hence the sequence is bounded if and only if it is eventually constant, which is if and only if $a_{n}$ is a perfect cube, for some $n \geqslant 0$. In particular, the sequence is bounded if $p$ is a perfect cube.
We now claim that, if $a_{n}$ is not a cube for some $n$, then neither is $a_{n+1}$. Indeed, if $a_{n}$ is not a cube, $q^{3}<a_{n}<(q+1)^{3}$ for some $q \in \mathbb{N}$, so that $c\left(a_{n}\right)=q^{3}$. Suppose to the contrary that $a_{n+1}$ is a cube. Then

$$
\begin{aligned}
a_{n+1} & =3 a_{n}-2 c\left(a_{n}\right)<3(q+1)^{3}-2 q^{3}=q^{3}+9 q^{2}+9 q+3 \\
& <q^{3}+9 q^{2}+27 q+27=(q+3)^{3} .
\end{aligned}
$$

Also, since $c\left(a_{n}\right)<a_{n}, a_{n+1}>a_{n}>q^{3}$, so $q^{3}<a_{n+1}<(q+3)^{3}$. It follows that the only possible values of $a_{n+1}$ are $(q+1)^{3}$ and $(q+2)^{3}$. However, in both of these cases,

$$
\begin{aligned}
& 3 a_{n}-2 q^{3}=a_{n+1}=(q+1)^{3} \Longleftrightarrow 3 a_{n}=3\left(q^{3}+q^{2}+1\right)+1 \\
& 3 a_{n}-2 q^{3}=a_{n+1}=(q+2)^{3} \Longleftrightarrow 3 a_{n}=3\left(q^{3}+2 q^{2}+4 q\right)+8
\end{aligned}
$$

a contradiction modulo 3 . This proves that, if $a_{n}$ is not a cube, then neither is $a_{n+1}$. Hence, if $p$ is not a perfect cube, $a_{n}$ is not a cube for any
$n \in \mathbb{N}$, and the sequence is not bounded. We conclude that the sequence is bounded if and only if $p$ is a perfect cube.
4. (a) Abby has a winning strategy for odd $N$ : Observe that, whenever any player writes down an odd number, the other player has to write down an even number. By adding 1 to that number, the first player can write down another odd number. Since Abby starts the game by writing down an odd number, she can force Brian to write down even numbers only. Since $N$ is odd, Abby will win the game. In particular, Abby has a winning strategy if $N=2011$.
(b) - Let $N=4 k$. If any player is forced to write down a number $m \in\{k+1, k+2, \ldots, 2 k\}$, the other player wins the game by writing down $2 m \in\{2 k+2,2 k+4, \ldots, 4 k\}$, for the players will have to write down the remaining numbers one after the other. Since there is an even number of numbers remaining, the latter player wins. This implies that the player who can write down $k$, i.e. has a winning strategy for $N=k$, wins the game for $N=4 k$.

- Similarly, let $N=4 k+2$. If any player is forced to write down a number $m \in\{k+1, k+2, \ldots, 2 k+1\}$, the other player wins the game by writing down $2 m \in\{2 k+2,2 k+4, \ldots, 4 k+2\}$, as in the previous case. Analogously, this implies that the player who has a winning strategy for $N=k$ wins the game for $N=4 k+2$.
Since Abby wins the game for $N=1,3$, while Brian wins the game for $N=2$, Brian wins the game for $N=8,10$ as well, and thus for $N=32,34,40,42$ too. Then Brian wins the game for a further 8 values of $N$ between 128 and 170, and thence for a further 16 values between 512 and 682 , and for no other values with $N \leqslant 2011$. Hence Brian has a winning strategy for precisely 31 values of N with $N \leqslant$ 2011.


## IMO Team Selection Test 1, June 2011

Problems

1. Find all pairs $(x, y)$ of integers that satisfy

$$
x^{2}+y^{2}+3^{3}=456 \sqrt{x-y} .
$$

2. We consider tilings of a rectangular $m \times n$-board with $1 \times 2$-tiles. The tiles can be placed either horizontally, or vertically, but they aren't allowed to overlap and to be placed partially outside of the board. All squares on the board must be covered by a tile.
(a) Prove that for every tiling of a $4 \times 2010$-board with $1 \times 2$-tiles there is a straight line cutting the board into two pieces such that every tile completely lies within one of the pieces.
(b) Prove that there exists a tiling of a $5 \times 2010$-board with $1 \times 2$-tiles such that there is no straight line cutting the board into two pieces such that every tile completely lies within one of the pieces.
3. The circles $\Gamma_{1}$ and $\Gamma_{2}$ intersect at $D$ and $P$. The common tangent line of the two circles closest to point $D$ touches $\Gamma_{1}$ in $A$ and $\Gamma_{2}$ in $B$. The line $A D$ intersects $\Gamma_{2}$ for the second time in $C$. Let $M$ be the middle of line segment $B C$.
Prove that $\angle D P M=\angle B D C$.
4. Determine all integers $n$ for which the polynomial $P(x)=3 x^{3}-n x-n-2$ can be written as the product of two non-constant polynomials with integer coefficients.
5. Let $A B C$ be a triangle with $|A B|>|B C|$. Let $D$ be the midpoint of $A C$. Let $E$ be the intersection of the angular bisector of $\angle A B C$ and the line $A C$. Let $F$ be the point on $B E$ such that $C F$ is perpendicular to $B E$. Finally, let $G$ be the intersection of $C F$ and $B D$.
Prove that $D F$ divides the line segment $E G$ into two equal parts.

## Solutions

1. Since the left hand side is an integer, the right hand side must also be an integer. The square root of an integer is either an integer or irrational (but never a rational number that is not an integer). Hence $\sqrt{x-y}$ must be an integer. The right hand side is divisible by 3 , so the left hand side must also be divisible by 3 . We deduce that $3 \mid x^{2}+y^{2}$. Since squares are always congruent to 0 or 1 modulo 3 , it follows that $x^{2} \equiv y^{2} \equiv 0 \bmod 3$, hence both $x$ and $y$ are divisible by 3 . Writing and substituting $x=3 a, y=3 b$ :

$$
9 a^{2}+9 b^{2}+3^{3}=456 \sqrt{3 a-3 b}
$$

Square roots of integers are either integer or irrational. But since $\sqrt{3 a-3 b}=$ $\frac{9 a^{2}+9 b^{2}+3^{3}}{456}$ is rational, it must be an integer as well. Hence $3 a-3 b$ is a square, and a multiple of 3 , so it must also be a multiple of 9 . Now we can divide both sides by 9 , which yields

$$
a^{2}+b^{2}+3=152 \sqrt{\frac{a-b}{3}} .
$$

Writing $a-b=3 c^{2}$ with $c \geqslant 0$, and substituting $a=b+3 c^{2}$ gives

$$
9 c^{4}+6 c^{2} b+2 b^{2}+3=152 c
$$

Considering this as a quadratic equation in $b$ and knowing that we are looking for real solutions, we can deduce that the discrimant must be nonnegative. Hence $36 c^{4}-8\left(9 c^{4}-152 c+3\right) \geqslant 0$ and thus $36 c^{4}+24 \leqslant 8 \cdot 152 c$. If $c \geqslant 4$ we have $36 c^{4}+24>36 \cdot 64 c \geqslant 8 \cdot 152 c$, this is a contradiction. Hence we may conclude $c \leqslant 3$. Furthermore, $152 c$ is even, and so is $6 c^{2} b+2 b^{2}$, hence $9 c^{4}+3$ must be even as well. We deduce that $c$ is odd, and consequently that the only possibilities are $c=1$ and $c=3$. For $c=3$ the discriminant is $36 \cdot 3^{4}-8\left(9 \cdot 3^{4}-152 \cdot 3+3\right) \equiv 1 \cdot 3(1-152) \equiv-1 \cdot 3 \cdot 151 \equiv 6 \bmod 9$. Hence it is not a square and the solutions of the quadratic equation will not be integers. Substituting $c=1$ gives

$$
9+6 b+2 b^{2}+3=152,
$$

or, equivalently,

$$
b^{2}+3 b-70=0
$$

This can also be written as $(b-7)(b+10)=0$. Hence either $b=7$ or $b=-10$. In the first case, we have $a=b+3 c^{2}=10$, so $x=30$ and $y=21$. In the second case, we have $a=b+3 c^{2}=-7$, hence $x=-21$ and $y=-30$. Note that both pairs do satisfy the equation. Hence the solutions are $(x, y)=(30,21)$ and $(x, y)=(-21,-30)$.

2a. Let a dividing line be a straight line that divides the board in two parts in such a way that every tile lies entirely in one of these parts. Suppose that there exists a tiling without dividing lines. Consider the columns $k$ and $k+1$, with $1 \leqslant k \leqslant 2009$. Then there is a tile lying horizontally in these two columns; otherwise, the vertical line between these two columns would have been a dividing one. There are $4 k$ squares in the first $k$ columns, an even number. Since every tile lying entirely inside the first $k$ columns, covers exactly 2 squares, it follows that the number of tiles lying horizontally in the columns $k$ and $k+1$ is even. We've shown earlier that this number is at least 1 , so it must be at least 2 .
So for every $k$ with $1 \leqslant k \leqslant 2009$, there are two tiles lying horizontally in columns $k$ and $k+1$. These tiles together cover $2 \cdot 2009 \cdot 2$ squares. Furthermore, for every $i$ with $1 \leqslant i \leqslant 3$, there must be a tile lying vertically in columns $i$ and $i+1$. These tiles together cover $3 \cdot 2$ squares. So the total number of squares covered by these tiles is $(2 \cdot 2009+3) \cdot 2>2 \cdot 2010 \cdot 2$. But the board only contains $4 \cdot 2010$ squares. This is a contradiction.
$\mathbf{2 b}$. We use induction on $n$ to show that for all $n \geqslant 3$, a $5 \times 2 n$-board can be tiled without dividing line. For $n=3$, this can be done as indicated in the picture. Suppose that we have a tiling of a $5 \times 2 n$-board without dividing lines. Then there are at least 4 tiles that lie vertically. Hence there is a $k$ with $1 \leqslant k \leqslant 2 n-1$
 such that there is a tile lying vertically in column $k$. Now add two columns, called $k_{1}$ and $k_{2}$, between columns $k$ and $k+1$. Every tile lying horizontally in columns $k$ and $k+1$, is replaced by two horizontal tiles, one in columns $k$ and $k_{1}$, one in columns $k_{2}$ and $k+1$. Since column $k$ contains a vertical tile, not all squares in columns $k_{1}$ and $k_{2}$ are covered yet. Also note that if in a certain row, the square in $k_{1}$ is not covered, then neither is the square in $k_{2}$ in the same row. Hence we can put a horizontal tile there. In that way, all the squares of the new columns are covered.
Now it's clear that no horizontal dividing lines are created in this process, and no vertical ones between columns that are unchanged. Moreover, there is at least one tile lying horizontally in columns $k_{1}$ and $k_{2}$, so there is no dividing line between these two columns. In the original board, there was no dividing line between columns $k$ and $k+1$. That implies that there are tiles lying horizontally in columns $k$ and $k_{1}$, and in columns $k_{2}$ and $k+1$. So there are no dividing lines between these pairs of columns. Hence the board constructed contains no dividing lines at all.
This completes the induction. We conclude that there is a tiling of a $5 \times 2010$-board without dividing line.
3. Let $S$ be the intersection of $P D$ and $A B$. Then $S$ lies on the radical axis of the two circles, so $|S A|=|S B|$. So $P S$ is a median in triangle $P A B$.
By the inscribed angle theorem on $\Gamma_{1}$ with chord $A P$, we have $\angle B A P=$ $180^{\circ}-\angle A D P=\angle C D P=\angle C B P$. By the inscribed angle theorem on $\Gamma_{2}$ with chord $B P$, we have $\angle A B P=\angle B C P$. Hence $\triangle P A B \sim \triangle P B C$ (aa). Since $P M$ is a median in triangle $P B C$, we have $\angle S P B=\angle M P C$. It follows that

$$
\begin{aligned}
\angle D P M & =\angle D P B+\angle B P M=\angle S P B+\angle B P M=\angle M P C+\angle B P M \\
& =\angle B P C=\angle B D C .
\end{aligned}
$$

4. Suppose that $P(x)$ can be written as $P(x)=A(x) B(x)$ with $A$ and $B$ are non-constant polynomials with integer coefficients. Since $A$ and $B$ are nonconstant, they both have degree at least 1 . The sum of the degrees is equal to the degree of $P$, hence equal to 3 . This implies that the two degrees are 1 and 2. So we can write, without loss of generality, that $A(x)=a x^{2}+b x+c$ and $B(x)=d x+e$, where $a, b, c, d$ and $e$ are integers. The product of the leading coefficients $a$ and $d$ is equal to the leading coefficient of $P$, hence equal to 3 . By multiplying both $A$ and $B$ if necessary, we may assume that both $a$ and $d$ are positive, so that they are equal to 1 and 3 in some order. First suppose that $d=1$. Substituting $x=-1$ yields

$$
P(-1)=3 \cdot(-1)^{3}+n-n-2=-5,
$$

so

$$
-5=P(-1)=A(-1) B(-1)=A(-1) \cdot(-1+e) .
$$

Note that $-1+e$ is a divisor of -5 , hence equal to $-5,-1,1$ or 5 , giving four possible values for $e$, namely $-4,0,2$ or 6 . Moreover, $x=-e$ is a zero of $B$, hence also of $P$.
If $e=-4$, then

$$
0=P(4)=3 \cdot 4^{3}-4 n-n-2=190-5 n,
$$

so $n=38$. So we can indeed factor $P(x)$ :

$$
3 x^{3}-38 x-40=\left(3 x^{2}+12 x+10\right)(x-4) .
$$

If $e=0$, then

$$
0=P(0)=-n-2,
$$

so $n=-2$. So we can indeed factor $P(x)$ :

$$
3 x^{3}+2 x=\left(3 x^{2}+2\right) x .
$$

If $e=2$, then

$$
0=P(-2)=3 \cdot(-2)^{3}+2 n-n-2=-26+n,
$$

so $n=26$. So we can indeed factor $P(x)$ :

$$
3 x^{3}-26 x-28=\left(3 x^{2}-6 x-14\right)(x+2) .
$$

If $e=6$, then

$$
0=P(-6)=3 \cdot(-6)^{3}+6 n-n-2=-650+5 n,
$$

so $n=130$. So we can indeed factor $P(x)$ :

$$
3 x^{3}-130 x-132=\left(3 x^{2}-18 x-22\right)(x+6) .
$$

Now suppose that $d=3$. Then we have:

$$
-5=P(-1)=A(-1) B(-1)=A(-1) \cdot(-3+e) .
$$

Note that $-3+e$ is a divisor of -5 , hence equal to $-5,-1,1$ or 5 . This gives four possible values for $e$, namely $-2,2,4$ or 8 . Furthermore, $x=\frac{-e}{3}$ is a zero of $B$, hence also of $P$. Note that $e$ is never divisible by 3 . We have

$$
0=P\left(\frac{-e}{3}\right)=3 \cdot\left(\frac{-e}{3}\right)^{3}+\frac{e}{3} n-n-2=-\frac{e^{3}}{9}+\frac{e-3}{3} n-2,
$$

so $\frac{e-3}{3} n=\frac{e^{3}}{9}+2$, hence $(e-3) n=\frac{e^{3}}{3}+6$. But this is a contradiction, since the left hand side is an integer, whereas the right hand side is not, since 3 does not divide $e$.

Finally, we deduce that the solutions are: $n=38, n=-2, n=26$ and $n=130$.
5. By $|A B|>|B C|$, the points $D$ and $E$ are distinct, and the order of the points on the line $A C$ is: $A, D, E, C$. Moreover, $\angle B C A>\angle C A B$, so $\angle B C A+\frac{1}{2} \angle A B C>90^{\circ}$, which implies that $F$ lies on the interior of $B E$. Consequently, $G$ lies on the interior of $B D$.
Now let $K$ be the intersection of $C F$ and $A B$. Since $B E$ is the angle bisector of $\angle A B C$, we have $\angle K B F=\angle F B C$. Furthermore, $\angle B F K=$ $90^{\circ}=\angle C F B$ and $|B F|=|B F|$, hence by (ASA), we have $\triangle K B F \cong$ $\triangle C B F$. This implies that $|B K|=|B C|$ and $|K F|=|C F|$. In particular, $F$ is the midpoint of $C K$. Since $D$ is the midpoint of $A C$, it follows that $D F$ is a midparallel in $\triangle A K C$, so $D F \| A K$ and $|D F|=\frac{1}{2}|A K|$.

From $D F \| A B$, it follows that $\triangle K G B \sim \triangle F G D$ (aa), hence

$$
\begin{equation*}
\frac{|D G|}{|B G|}=\frac{|F D|}{|K B|}=\frac{\frac{1}{2}|A K|}{|K B|}=\frac{\frac{1}{2}(|A B|-|K B|)}{|K B|}=\frac{\frac{1}{2}(|A B|-|B C|)}{|B C|} . \tag{3}
\end{equation*}
$$

The angle bisector theorem now implies that $\frac{|A E|}{|C E|}=\frac{|A B|}{|C B|}$, or, equivalently $\frac{|A C|-|C E|}{|C E|}=\frac{|A B|}{|B C|}$. This yields $|B C| \cdot(|A C|-|C E|)=|A B| \cdot|C E|$, so $|B C| \cdot|A C|=|C E| \cdot(|A B|+|B C|)$, hence

$$
|C E|=\frac{|B C| \cdot|A C|}{|A B|+|B C|}
$$

Continuing the calculation:

$$
\begin{aligned}
|D E| & =|D C|-|C E|=\frac{1}{2}|A C|-|C E| \\
& =\frac{\frac{1}{2}|A C| \cdot(|A B|+|B C|)-|A C| \cdot|B C|}{|A B|+|B C|}=\frac{\frac{1}{2}|A C| \cdot(|A B|-|B C|)}{|A B|+|B C|} .
\end{aligned}
$$

Thus

$$
\frac{|D E|}{|C E|}=\frac{\frac{\frac{1}{2}|A C| \cdot(|A B|-|B C|)}{|A B|+|B C|}}{\frac{|B C| \cdot|A C|}{|A B|+|B C|}}=\frac{\frac{1}{2}(|A B|-|B C|)}{|B C|} .
$$

Combined with (3) this gives

$$
\frac{|D E|}{|C E|}=\frac{|D G|}{|B G|} .
$$

In triangle $D B C$ this implies that $E G \| B C$. Now let $S$ be the intersection of $D F$ and $E G$. Then we have

$$
\angle S G F=\angle E G F=\angle F C B=\angle F K B=\angle G F D=\angle G F S,
$$

so $\triangle S F G$ is isosceles, with $|S F|=|S G|$. Moreover,

$$
\angle S E F=\angle G E F=90^{\circ}-\angle E G F=90^{\circ}-\angle G F S=\angle S F E,
$$

so $\triangle S F E$ is also isosceles, with $|S F|=|S E|$. We deduce that $|S G|=|S E|$, hence that $S$ is the midpoint of $E G$.

## IMO Team Selection Test 2, June 2011

## Problems

1. Let $n \geqslant 2$ and $k \geqslant 1$ be positive integers. In a country there are $n$ cities and between each pair of cities there is a bus connection in both directions. Let $A$ and $B$ be two different cities. Prove that the number of ways in which you can travel from $A$ to $B$ by using exactly $k$ buses is equal to

$$
\frac{(n-1)^{k}-(-1)^{k}}{n} .
$$

2. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$
x f(x+x y)=x f(x)+f\left(x^{2}\right) f(y)
$$

for all $x, y \in \mathbb{R}$.
3. Let $\Gamma_{1}$ and $\Gamma_{2}$ be two intersecting circles with midpoints respectively $O_{1}$ and $O_{2}$, such that $\Gamma_{2}$ intersects the line segment $O_{1} O_{2}$ in a point $A$. The intersection points of $\Gamma_{1}$ and $\Gamma_{2}$ are $C$ and $D$. The line $A D$ intersects $\Gamma_{1}$ a second time in $S$. The line $C S$ intersects $O_{1} O_{2}$ in $F$. Let $\Gamma_{3}$ be the circumcircle of triangle $A D F$. Let $E$ be the second intersection point of $\Gamma_{1}$ and $\Gamma_{3}$.
Prove that $O_{1} E$ is tangent to $\Gamma_{3}$.
4. Prove that there exists no infinite sequence of prime numbers $p_{0}, p_{1}, p_{2}, \ldots$ such that for all positive integers $k$ :

$$
p_{k}=2 p_{k-1}+1 \quad \text { or } \quad p_{k}=2 p_{k-1}-1 .
$$

5. Find all triples $(a, b, c)$ of positive integers with $a+b+c=10$ such that there are $a$ red, $b$ blue and $c$ green points (all different) in the plane satisfying the following properties:

- for each red point and each blue point we consider the distance between these two points; the sum of these distances is 37 ;
- for each green point and each red point we consider the distance between these two points; the sum of these distances is 30 ;
- for each blue point and each green point we consider the distance between these two points; the sum of these distances is 1 .


## Solutions

1. Let $\alpha(k)$ be the number of ways to travel from city $A$ to city $B \neq A$ by using $k$ buses. Let $\beta(k)$ be the number of ways to travel from city $A$ to city $A$ by using $k$ buses. If we start in city $A$ and then take a bus $k$ times, this can be done in $(n-1)^{k}$ ways. In $\beta(k)$ of the cases we end at city $A$ and in $(n-1) \alpha(k)$ of these cases we end at a city different from $A$. Hence

$$
\begin{equation*}
(n-1) \alpha(k)+\beta(k)=(n-1)^{k} . \tag{4}
\end{equation*}
$$

Now suppose that $k \geqslant 2$. To travel from city $A$ to city $A$ by using exactly $k$ buses, we take a bus from $A$ to an arbitrary city (this can be done in ( $n-1$ ) ways); then we must travel from a city different from $A$ to city $A$ by using $k-1$ buses, which can be done in $\alpha(k-1)$ ways. Therefore

$$
\begin{equation*}
\beta(k)=(n-1) \alpha(k-1) \quad \text { for } k \geqslant 2 . \tag{5}
\end{equation*}
$$

Now we substitute this expression for $\beta(k)$ in (4). This yields for $k \geqslant 2$ that

$$
(n-1) \alpha(k)+(n-1) \alpha(k-1)=(n-1)^{k}
$$

and hence

$$
\begin{equation*}
\alpha(k)=(n-1)^{k-1}-\alpha(k-1) . \tag{6}
\end{equation*}
$$

Now we prove by induction to $k$ that for $n \geqslant 2$ and $k \geqslant 1$ :

$$
\alpha(k)=\frac{(n-1)^{k}-(-1)^{k}}{n}
$$

For $k=1$ this is equivalent to $\alpha(1)=\frac{(n-1)+1}{n}=1$ and that is true, because there is exactly one way to travel from city $A$ to city $B \neq A$ by using one bus. Now let $m \geqslant 1$ be an integer and suppose that the expression for $\alpha(k)$ has been proven for $k=m$. Then, using (6), for $k=m+1 \geqslant 2$ :

$$
\begin{aligned}
\alpha(m+1) & =(n-1)^{m}-\alpha(m)=(n-1)^{m}-\frac{(n-1)^{m}-(-1)^{m}}{n} \\
& =\frac{n(n-1)^{m}-(n-1)^{m}+(-1)^{m}}{n}=\frac{(n-1)^{m+1}-(-1)^{m+1}}{n}
\end{aligned}
$$

and that is exactly the expression we wanted to prove for $k=m+1$. This completes the induction.
2. Substituting $x=0$ and $y=0$ yields $0=f(0)^{2}$, hence $f(0)=0$. Substituting $x=1$ and $y=-1$ yields $f(0)=f(1)+f(1) f(-1)$, hence $0=f(1)(1+f(-1))$, therefore $f(1)=0$ or $f(-1)=-1$. Substituting $x=-1$ yields

$$
\begin{equation*}
-f(-1-y)=-f(-1)+f(1) f(y) \quad \text { for all } y \in \mathbb{R} \tag{7}
\end{equation*}
$$

Suppose that $f(1)=0$, then this is equivalent to $-f(-1-y)=-f(-1)$ and because $-1-y$ takes all values in $\mathbb{R}$, this means that $f$ is constant. Because $f(0)=0$, it must be true that $f(x)=0$ for all $x$. It is clear that this function satisfies the original equation. We found our first solution: $f(x)=0$ for all $x \in \mathbb{R}$.
Now suppose that $f(1) \neq 0$, then $f(-1)=-1$. Now substitute $y=-1$ in (7), then we get $-f(0)=-f(-1)+f(1) f(-1)$, hence $0=1-f(1)$, therefore $f(1)=1$. Substituting $x=1$ in the original equation yields

$$
\begin{equation*}
f(1+y)=1+f(y) \quad \text { for all } y \in \mathbb{R} \tag{8}
\end{equation*}
$$

Furthermore substituting $y=-1$ in the original equation yields $x f(0)=$ $x f(x)-f\left(x^{2}\right)$, hence

$$
\begin{equation*}
x f(x)=f\left(x^{2}\right) \quad \text { for all } x \in \mathbb{R} . \tag{9}
\end{equation*}
$$

The original equation now can be written as

$$
x f(x+x y)=x f(x)+x f(x) f(y)=x f(x)(1+f(y)) .
$$

If $x \neq 0$, we may divide the left and right hand side by $x$ and this yields together with (8):

$$
f(x+x y)=f(x) f(1+y) \quad \text { for } x \neq 0 .
$$

Notice that this is also true for $x=0$. Now define $z=1+y$. Because this takes all values in $\mathbb{R}$, we get

$$
f(x z)=f(x) f(z) \quad \text { for all } x, z \in \mathbb{R} .
$$

Applying this to (9), we find

$$
x f(x)=f\left(x^{2}\right)=f(x) f(x)
$$

hence for all $x$ we have $f(x)=0$ or $f(x)=x$. Now suppose there is a $x \neq 0$ with $f(x)=0$, then

$$
1=f(1)=f\left(x \cdot \frac{1}{x}\right)=f(x) f\left(\frac{1}{x}\right)=0,
$$

which is a contradiction. Hence for all $x \neq 0$ we have $f(x)=x$. This is also true for $x=0$. Substituting it in the original equation shows that this is indeed a solution. The two functions satisfying the condition are $f(x)=0$ for all $x \in \mathbb{R}$ and $f(x)=x$ for all $x \in \mathbb{R}$.
3. The intersection point of $O_{1} O_{2}$ and the arch $S D$ of $\Gamma_{1}$ containing $C$, is named $T$. Because $A$ lies in the interior of $\Gamma_{1}$, we now know

$$
\begin{aligned}
\angle O_{1} A S & =\angle A T S+\angle T S A \quad \quad \text { exterior angle theorem in } \triangle A T S \\
& =\angle O_{1} T S+\angle T S D \\
& =\angle T S O_{1}+\angle T S D \quad \triangle O_{1} S T \text { is isosceles }\left(\left|O_{1} S\right|=\left|O_{1} T\right|\right) .
\end{aligned}
$$

The line $O_{1} O_{2}$ is perpendicular to $C D$ and splits the line segment $C D$ into two equal parts, hence it is the perpendicular bisector of $C D$. Therefore $T$ lies on the perpendicular bisector of $C D$, which yields that arches $T C$ and $T D$ are equally long. Hence by the inscribed angle theorem $\angle T S D=$ $\angle C S T$. Thus

$$
\angle O_{1} A S=\angle T S O_{1}+\angle T S D=\angle T S O_{1}+\angle C S T=\angle C S O_{1}=\angle F S O_{1}
$$

This means that $\triangle O_{1} A S \sim \triangle O_{1} S F(\mathrm{hh})$. This yields

$$
\frac{\left|O_{1} A\right|}{\left|O_{1} S\right|}=\frac{\left|O_{1} S\right|}{\left|O_{1} F\right|},
$$

hence $\left|O_{1} A\right| \cdot\left|O_{1} F\right|=\left|O_{1} S\right|^{2}=\left|O_{1} E\right|^{2}$. Because $A$ and $F$ lie on the same side of $O_{1}$, it is even true that $O_{1} A \cdot O_{1} F=O_{1} E^{2}$. Using the power theorem we now see that $O_{1} E$ is tangent to the circumcircle of $\triangle A F E$ and that is $\Gamma_{3}$.
4. Suppose that such an infinite sequence exists. By eventually leaving out the first two elements, we can make sure that the first prime number of the sequence is at least 5 . Now we assume without loss of generality that $p_{0} \geqslant 5$. Then we know that $p_{0} \not \equiv 0 \bmod 3$.
Suppose that $p_{0} \equiv 1 \bmod 3$. We prove by induction to $k$ that for all positive integers $k: p_{k} \equiv 1 \bmod 3$ and $p_{k}=2 p_{k-1}-1$. Namely, suppose that $k \geqslant 1$ and $p_{k-1} \equiv 1 \bmod 3$. Then $2 p_{k-1} \equiv 2 \bmod 3$, hence $2 p_{k-1}+1$ is divisible by 3 . Because $p_{k}$ is a prime number greater than 3 , we have $p_{k}=2 p_{k-1}-1$ and $p_{k} \equiv 2-1 \equiv 1 \bmod 3$. This completes the induction. In the case $p_{0} \equiv 2 \bmod 3$, we can prove analogously that $p_{k} \equiv 2 \bmod 3$ and $p_{k}=2 p_{k-1}+1$ for all positive integers $k$.
Now we can compose a direct formula for the sequence. If $p_{0} \equiv 1 \bmod 3$, then we get $p_{k}=\left(p_{0}-1\right) 2^{k}+1$ for all $k \geqslant 0$. If $p_{0} \equiv 2 \bmod 3$, then we get $p_{k}=\left(p_{0}+1\right) 2^{k}-1$. We can prove these formulas by induction again.
By Fermat's little theorem $2^{p_{0}-1} \equiv 1 \bmod p_{0}$, hence also $\left(-p_{0}+1\right) 2^{p_{0}-1} \equiv$ $1 \bmod p_{0}$ and $\left(p_{0}+1\right) 2^{p_{0}-1} \equiv 1 \bmod p_{0}$. This yields $p_{0} \mid\left(p_{0}-1\right) 2^{p_{0}-1}+1$
and $p_{0} \mid\left(p_{0}+1\right) 2^{p_{0}-1}-1$. We see that $p_{0}$ always is a divisor of $p_{p_{0}-1}$. Because it is also clear that $p_{p_{0}-1}$ is greater than $p_{0}$, this yields that $p_{p_{0}-1}$ is not a prime number. This is a contradiction.
5. We construct triangles consisting of a blue, a red and a green point. These triangles may also be degenerated. In each of these triangles the (nonstrict) triangle inequality holds: the distance between the blue and the red point is at most the sum of the distances between the blue and the green and between the red and the green point. We add all these triangle inequalities (one for each triangle we can form with a blue, a red and a green point). We now count each distance between a red and a blue point $c$ times (because with a fixed blue and red point you may choose $c$ points as third green point), each distance between a green and a red point $b$ times and each distance between a blue and a green point $a$ times. Thus, we get

$$
37 c \leqslant 30 b+a .
$$

Because $a+b+c=10$, this yields $37 c \leqslant 30 b+(10-b-c)=10+29 b-c$, hence $38 c \leqslant 10+29 b$. Otherwise stated

$$
\begin{equation*}
\frac{38 c-10}{29} \leqslant b \tag{10}
\end{equation*}
$$

We can also apply the triangle inequality differently: the distance between the green and the red point is at most the sum of the distances between the red and the blue and between the blue and the green point. If we add these inequalities, we get

$$
30 b \leqslant 37 c+a .
$$

This yields $30 b \leqslant 37 c+(10-b-c)=10+36 c-b$, so $31 b \leqslant 10+36 c$, otherwise stated

$$
\begin{equation*}
b \leqslant \frac{10+36 c}{31} \tag{11}
\end{equation*}
$$

Combining (10) and (11) yields

$$
\frac{38 c-10}{29} \leqslant \frac{10+36 c}{31}
$$

hence $31(38 c-10) \leqslant 29(10+36 c)$. Expanding yields $134 c \leqslant 600$, therefore $c \leqslant 4$. Now we consider one by one all possibilities for $c$.
Suppose $c=1$. Then (11) yields $b \leqslant \frac{46}{31}<2$, which yields $b=1$. Now we get $(a, b, c)=(8,1,1)$.
Suppose $c=2$. Then (10) and (11) yield $2<\frac{66}{29} \leqslant b \leqslant \frac{82}{31}<3$, hence there is no integer $b$ that satisfies the conditions.

Suppose $c=3$. Then (10) and (11) yield $3<\frac{104}{29} \leqslant b \leqslant \frac{118}{31}<4$, hence there is no integer $b$ that satisfies the conditions.
Suppose $c=4$. Then (10) and (11) yield $4<\frac{142}{29} \leqslant b \leqslant \frac{154}{31}<5$, hence there is no integer $b$ that satisfied the conditions.

The only triple that possibly could satisfy the conditions is $(8,1,1)$. Now we show that there are indeed 8 red points, 1 blue point and 1 green point in the plane that satisfy all conditions. We use a standard coordinate system. Choose for the blue point $(0,0)$ and for the green point $(1,0)$. Choose red points $(i, 0)$ with $2 \leqslant i \leqslant 8$. Also choose a red point such that that point together with the green and blue point forms an isosceles triangle with two sides of length 2 . The sum of the distances between the red points and the blue point now is $2+2+3+\cdots+8=37$. The sum of the distances between the red points and the green point is $2+1+2+\cdots+7=30$. The distance between the blue point and the green point is 1 .
We conclude that the only possible solution is: $(8,1,1)$.

## Junior Mathematical Olympiad, October 2010

## Problems

## Part 1

1. The letters $A, B, C$ and $D$ represent digits. The following holds:

$$
\begin{array}{cc}
A & B \\
C & A \\
\hline D & A
\end{array} \text { and } \begin{array}{ccc}
A & B & \\
C & A & - \\
\hline & & A
\end{array}
$$

What is $D$ ?
A) 5
B) 6
C) 7
D) 8
E) 9
2. Peter is constructing a sequence of seven integers (they can be negative or zero as well), such that the sum of four consecutive integers is always 1. He wants his sequence to contain as many integers as possible, which are greater than 13 . What is the maximal number of such integers?
A) 0
B) 2
C) 3
D) 5
E) 6
3. An ant is walking on the surface of a cuboid with edges of length 3,4 and 5. It starts walking on a vertex, and it wants to visit all the other seven vertices. It doesn't need to return to its starting vertex. What is the length of the shortest possible route that accomplishes this?
A) 24
B) 25
C) 26
D) 27
E) 28
4. The integers 1 up to 5 are placed to form a circle. We add every pair of neighbours together. The five sums that we obtain, turn out to be consecutive integers. What are the neighbours of 1 ?

A) 2 and 4
B) 2 and 5
C) 3 and 4
D) 3 and 5
E) 4 and 5
5. Bert and Ernie both have 64 candies. Every day, one of them gives half of his candies to the other. After six days, Bert has 61 candies, so Ernie has 67 candies. How many of the six days has Ernie shared his candies?
A) 1
B) 2
C) 3
D) 4
E) 5
6. If you add 36 to 37 , you obtain 73 . When its digits are written in reverse order, this becomes 37 again. How many two-digit integers are there with the property that if you add 36 to it, then write its digits in reverse order, you obtain the integer you started with?
A) 4
B) 5
C) 6
D) 9
E) 10
7. One hundred students participate in a mathematical olympiad. Problem 1 was solved by 90 participants. Problem 2 was solved by 80 participants and problem 3 has been solved by 75 participants. What is the minimal number of participants that solved all of the problems?
A) 35
B) 45
C) 54
D) 55
E) 60
8. From 125 small cubes, a $5 \times 5 \times 5$-cube is made. In every direction, the cubes are coloured white and black alternatingly; the cubes on the vertices are black. We only consider the cubes that are visible from the outside. What is the difference between the number of black cubes and the number of white ones?
A) 2 more black cubes
B) 1 more black cube
C) no difference
D) 1 more white cube
E) 2 more white cubes
9. We start by drawing an equilateral triangle, and then its circumcircle. Around this circle, we draw a perfectly fitting square. We then draw its circumcircle, and draw a perfectly fitting pentagon around it, and so on, up to a regular 16 -gon. In the figure, you can see that the area inside of the pentagon is divided in 17 pieces. In how many pieces is the area inside the
 regular 16 -gon divided?
A) 134
B) 136
C) 248
D) 264
E) 267
10. How many integers between 1 and 1000 do not contain the digit 1 ?
A) 700
B) 728
C) 729
D) 880
E) 900
11. A square with edges of length 2010 is divided into nine rectangles by four lines which are parallel to the edges. You can see an example in the figure. You can divide this square in such a way that the perimeter of the rectangles obtained are nine consecutive integers. What is the perimeter of the largest rectangle?

A) 671
B) 1340
C) 1790
D) 2684
E) 3577
12. A farmer has a stack of hay to feed his horse, cow and goat. With it, he can feed his cow and horse for 12 months, or his cow and goat for 15 months, or his horse and goat for 20 months. For how many months can he feed all three of his animals?
A) $7 \frac{5}{6}$
B) 9
C) 10
D) $15 \frac{2}{3}$
E) 47
13. There are twenty-five towers: five of height 1 , five of height 2 , five of height 3 , five of height 4 and five of height 5 . The towers have to be places on a $5 \times 5-$ board with on each square a tower, such that in every row and every column, every height occurs exactly once. Moreover, in the direction of each of the arrows in the figure, you have to be able to see exactly the number of towers mentioned at that arrow. You
 can't see a tower if it's behind a higher tower. What number should be filled in the place of the question mark?
A) 1
B) 2
C) 3
D) 4
E) 5
14. Tania has played 10 basketball matches this season. In the sixth up to the ninth match, she scored $23,14,11$ and 20 points, respectively. As a result, her average score per match became higher after nine matches, than it was after five matches. After ten matches, she has an average score of more than 18 points. What is the minimal number of points that Tania scored in the tenth match?
A) 19
B) 27
C) 28
D) 29
E) 31
15. Lucas' sequence starts with the integers 1 and 3 . After that, the next integer in the sequence is found by adding the two preceding integers. So you obtain $1,3,4,7,11, \ldots$ What is the last digit of the hundredth integer in Lucas' sequence?
A) 1
B) 3
C) 4
D) 7
E) 8

## Part 2

The answer to each problem is a number.

1. The length of the side of the small square is 5 , and the length of the side of the large square is 10 . What is the total area of all the black pieces?
2. The following sum is incorrect:


$$
\begin{array}{cccccccc} 
& 7 & 4 & 2 & 5 & 8 & 6 & \\
& 8 & 2 & 9 & 4 & 3 & 0 & + \\
\hline 1 & 2 & 1 & 2 & 0 & 1 & 6 &
\end{array}
$$

This sum can made correct by picking two distinct digits and replace every occurrence of one of them with the other one. Which two digits should you pick?
3. A rhombus is cut from a pattern of grey and white squares. What fraction of the rhombus is grey?

4. Consider all four-digit integer in which each of the digits $1,2,3$ and 4 occur exactly once. What's the average of all these integers?
5. Three equilateral triangles with edges of length 21 are placed on a white table in such a way that we obtain an equilateral triangle with edges of length 36. In the centre, a small triangular piece of table is visible. How long are the edges of that piece?

6. A rectangle is divided into eight squares. The area of the grey square is 1 . What is the area of the rectangle?

7. Anton and Ben are competing in a race. Ben runs three times as fast as Anton, but Anton gets a 30 meter lead. The arrive simultaneously at the finish line. What is the distance in meters that Ben has run when he crosses the finish line?
8. A square is divided into seven pieces by its diagonal and by two lines parallel to the edges. Of three pieces, the area is given. What is the area of the square?

9. Anne answered 33 questions, most of them well and the rest even very well. Both grades are rewarded with a fixed number of points per questions, very well answered questions being rewarded with more points than well answered questions. Both number of points are integers $1,2,3, \ldots$, or 10, but we don't know exactly which two integers they are. When Anne calculated her average number of points per question, it turned out to be an integer. How many questions did Anne answer very well?
10. On a circle, there are eleven points, numbered from 1 up to 11 . From each point except point 11, a line segment is drawn between that point, and the point with the next number. Another line segment is drawn between point 11 and point 1 . What is the maximal number of intersections of these 11 line segments? In the figure, there's an example with only 16 intersections.


## Solutions

## Part 1

1. E) 9
2. B) 5
3. D) 2684
4. E) 6
5. B) 45
6. C) 10
7. 

B) 25
8. A) 2 more black
13. B) 2
4. C) 3 and 4
9. C) 248
14. D) 29
5. D) 4
10. B) 728
15. D) 7

## Part 2

1. 25
2. 2 and 6
3. $\frac{2}{3}$
4. $2777 \frac{1}{2}$
5. 9
6. $\frac{595}{8}=74 \frac{3}{8}$
7. 45
8. 484
9. 11
10. 44

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 INSTRUMENTS

