

Second round

Dutch Mathematical Olympiad



Friday 17 March 2017

Solutions

B-problems

- B1.** 11 Of every three consecutive integers exactly one is a multiple of three. Consider a sequence of k consecutive integers. If k itself is a multiple of three, say $k = 3\ell$, then the sequence contains exactly ℓ multiples of three: one from each group of three consecutive numbers. If k is not a multiple of three, say $k = 3\ell + 1$ or $k = 3\ell + 2$, then the sequence contains at least ℓ multiples of three since $k \geq 3\ell$. In each case we have $3\ell \geq k - 2$ so that the number of multiples of three is at least $\ell \geq \frac{k-2}{3}$.

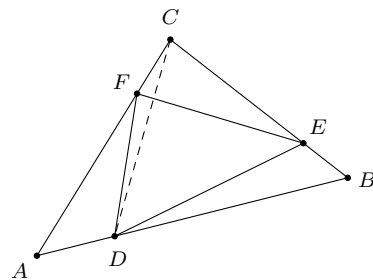
Of every five consecutive integers exactly one is a multiple of five. Consider a sequence of k consecutive integers. If k is a multiple of five, say $k = 5m$, then the sequence contains exactly m multiples of five. If k is not a multiple of five, say $k = 5m - 1$, $k = 5m - 2$, $k = 5m - 3$, or $k = 5m - 4$, then the sequence contains at most m multiples of five. In each case $5m \leq k + 4$ and hence the number of multiples of five is at most $m \leq \frac{k+4}{5}$.

If a balanced sequence has length k , then the above considerations show that $\frac{k-2}{3} \leq \frac{k+4}{5}$. Multiplying both sides by 15 yields $5k - 10 \leq 3k + 12$. This implies that $2k \leq 22$ and therefore that $k \leq 11$.

The sequence $10, 11, \dots, 20$ has length 11 and is balanced. Indeed, it contains 3 multiples of three and 3 multiples of five. We conclude that $k = 11$ is the maximal length of a balanced sequence.

- B2.** $\frac{35}{2}$ First, we will determine the area of triangle BDE . As an intermediate step, we will compute the area of triangle BCD . From the data provided in the problem we deduce that $|BD| = \frac{3}{4} \cdot |AB|$ and $|BE| = \frac{1}{4} \cdot |BC|$.

Comparing the base AB of triangle ABC to the base BD of triangle BCD , we observe that $|BD|$ is a factor $\frac{3}{4}$ larger than $|AB|$, while the two triangles have equal heights with respect to these bases. Hence, the area of triangle BCD is $\frac{3}{4} \cdot 40 = 30$.



We will apply the same idea to triangles BCD and BDE . The base BE of triangle BDE is $\frac{1}{4}$ times as long as the base BC of triangle BCD . The heights of the triangles with respect to these bases are equal. Hence, the area of triangle BDE is $\frac{1}{4} \cdot 30 = \frac{15}{2}$.

In a similar way, we can deduce that the areas of triangles CEF and ADF equal $\frac{15}{2}$. The area of triangle DEF is therefore equal to $40 - 3 \cdot \frac{15}{2} = \frac{35}{2}$.

B3. Multiple sol. One possible sequence is the following:

0 1 2 2 4 4 4 4 8 8 8 8 8 8 8 8

The sequences written by the second through sixth student are the following:

1 1 2 2 4 4 4 4 8 8 8 8 8 8 8 8
 2 2 2 2 4 4 4 4 8 8 8 8 8 8 8 8
 4 4 4 4 4 4 4 4 8 8 8 8 8 8 8 8
 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8
 16 16 16 16 16 16 16 16 16 16 16 16 16 16 16 16

The seventh student will then write the same sequence as the sixth student did.

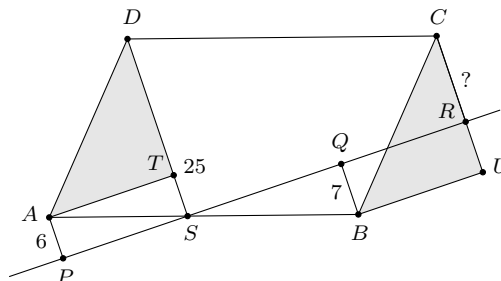
There are also other starting sequences that are a solution to this problem. In each correct solution, the second sequence must consist of twice the number 1, twice the number 2, four times the number 4, and eight times the number 8.

B4. 12 We will show that:

$$|AP| + |BQ| + |CR| = |DS|.$$

From this we will derive the requested length:
 $|CR| = |DS| - |AP| - |BQ| = 25 - 6 - 7 = 12.$

Let T be the foot of the perpendicular from A on the line DS . Since $APST$ is a rectangle, we have $|ST| = |AP|.$



It remains to be shown that $|DT| = |BQ| + |CR|.$ Let U be the foot of the perpendicular from B on the line CR . Since $BURQ$ is a rectangle, we have $|RU| = |BQ|$ and therefore $|CU| = |BQ| + |CR|.$ Triangles BUC and ATD are similar since the sides of the two triangles are pairwise parallel. Also the two triangles have equal size, since $|AD| = |BC|.$ It follows that $|DT| = |CU|$ and therefore that $|DT| = |BQ| + |CR|.$

B5. 500 First, consider the blue block number 2017. All yellow blocks must lie to the left of it. This includes the yellow block number 1. To the right of yellow block number 1 there can be at most 1 blue block; this must be blue block 2017. We see that there are no blue blocks to the right of block block 2017. Hence, it is the rightmost block of the row.

Now consider the blue block number 2016. It is to the left of yellow block 1 (as we have seen). There must be at least 2016 yellow blocks to the left of blue block 2016. These must therefore be the yellow blocks 2 to 2017. It follows that the second block from the right must be the yellow block 1.

Now consider the yellow block number 2. It must be to the left of the blue blocks 2016 and 2017. In turn, all other blue blocks must be to the left of the yellow block 2. Hence, the third block from the right is the blue block 2016.

Continuing this line of reasoning, we see that the only possible way to arrange the blocks is as follows:

$$Y_{2017}, B_1, Y_{2016}, B_2, \dots, Y_2, B_{2016}, Y_1, B_{2017}.$$

The 1000th block from the left is therefore the blue block number 500.

C-problems

- C1.** (a) A matching combination is 'BABC' having a top row and bottom row equal to 1101011101. *All other solutions are of the form 'BABC BABC... BABC'.*

- (b) Suppose that we have a matching combination using only tiles of types B, C, and D.

The top row of these tiles always starts with a 1. In a matching combination, the bottom row must therefore start with a 1 as well. This rules out the first tile being of type D. Type C is ruled out as well, since otherwise the second tile must have a top row starting with 0. The first tile of a matching combination must therefore be of type B.

The next tile must have a bottom row starting with 0. Hence, it must be of type D. After that, we again require a tile with a bottom row starting with a 0. That is, we again require a tile of type D. This continues indefinitely. After depleting our supply of 1000 tiles of type D, we still need another tile having a bottom row starting with a 0. It follows that we cannot complete our row to a matching combination: a contradiction.

We conclude that there is no matching combination using only tiles of types B, C, and D. *The proof of part (c) will give an alternative proof for part (b).*

- (c) First, consider the number of 1's in the top row and bottom row of each type of tile. Tiles of type B, D, and E have equal numbers of 1's in the top and bottom row. Tiles of type C have more 1's in the bottom row than in the top row. Since a matching combination must have equal numbers of 1's in both rows, it cannot contain tiles of type C.

A matching combination can therefore only contain tiles of types B, D, and E. Tiles of type D have an equal number of digits in the top row and bottom row (three digits). In contrast, tiles of type B and E have more digits in their top row than in their bottom row. Since any matching combination has equal numbers of digits in both rows, it can contain tiles of neither type B nor type E.

A matching combination can therefore only contain tiles of type D. But that is also not possible since tiles of type D have a top row starting with 1 and a bottom row starting with 0. We conclude that no matching combination exists containing only tiles of types B, C, D, and E.

- C2.** (a) The two-digit squares are 16, 25, 36, 49, 64, and 81. We consider all cases based on the possibilities for the first and last digit.

No two-digit square ends in a 2, 3, 7, or 8. Also, no two-digit square starts with a 5 or 9. Hence, we only need to consider multi-squares with first and last digit equal to 1, 4, or 6.

In the case that the first and last digit are 1, the first square must be 16 and the second square must be 81. This yields the multi-square 1681. In the case that the first and last digit equal 4, the first square must be 49 and the second square must be 64. This yields the multi-square 4964. In the case that the first and last digit equal 6, the first square must be 64 and the second square must be 16 or 36. This yields two multi-squares: 6416 and 6436. In total, there are four multi-squares having the same first and last digit: 1681, 4964, 6416, and 6436.

- (b) Let K be a six-digit multi-square, say $K = \overline{abcdef}$, where $a, b, c, d, e,$ and f are the six digits of K . The fact that K is a multi-square means that \overline{ab} , \overline{cd} , and \overline{ef} are two-digit squares, hence equal to 16, 25, 36, 49, 64, or 81. We want K to be a square, say $K = n^2$. To determine all solutions, we consider the different cases for \overline{ab} .

$\overline{ab} = 16$ Since $K > 160000$, we must have $n > 400$. Write $n = 400 + x$, where x is a positive integer. We have $K = (400 + x)^2 = 160000 + 800x + x^2$. Since $K < 170000$, it follows that $800x < 10000$ and therefore $x \leq 12$.

Observe that the last two digits \overline{ef} of $K = 160000 + 800x + x^2$ are equal to the last two digits of x^2 . Since the last two digits of $10^2 = 100$, $11^2 = 121$, and $12^2 = 144$ do not form a square, the candidates $x = 10, 11, 12$ are ruled out. Since $1^2 = 1$, $2^2 = 4$, and $3^2 = 9$ have only one digit, also the candidates $x = 1, 2, 3$ are ruled out.

We consider the remaining candidates $x = 4, 5, 6, 7, 8, 9$. In these cases, x^2 is a two-digit number, which implies that $\overline{cd} = 8 \cdot x$. This is a square only in the case $x = 8$. Hence, we find one solution: $408^2 = 166464$.

$\overline{ab} = 25$ Similarly to the previous case, we can write $n = 500 + x$, where x is a positive integer. We have $K = (500 + x)^2 = 250000 + 1000x + x^2$. Since $K < 260000$, it follows that $1000x < 10000$ and hence $x \leq 9$.

We observe that for every possible choice of x , the digit d of K will be equal to 0. Hence, there are no solutions.

$\overline{ab} = 36$ We write $n = 600 + x$, where x is a positive integer. We have $K = (600 + x)^2 = 360000 + 1200x + x^2$. Since $K < 370000$, it follows that $1200x < 10000$ and hence $x \leq 8$. Also, we have $x \geq 4$ as otherwise digit e will be equal to 0.

For the remaining candidates $x = 4, 5, 6, 7, 8$ we obtain $\overline{cd} = 12 \cdot x$. However, this is not a square for any of the possible choices for x . Hence, there are no solutions.

$\overline{ab} = 49$ We write $n = 700 + x$, where x is a positive integer. We have $K = (700 + x)^2 = 490000 + 1400x + x^2$. Since $K < 500000$, it follows that $1400x < 10000$ and hence $x \leq 7$. As in the previous case, we also have $x \geq 4$ because digit e cannot be 0.

For the remaining candidates $x = 4, 5, 6, 7$ we see that $\overline{cd} = 14 \cdot x$ is not a square. Hence, there are no solutions.

$\overline{ab} = 64$ We write $n = 800 + x$, where x is a positive integer. We have $K = (800 + x)^2 = 640000 + 1600x + x^2$. Since $K < 650000$, it follows that $1600x < 10000$ and hence $x \leq 6$. Again, we also have $x \geq 4$.

Of the remaining candidates $x = 4, 5, 6$, the number $\overline{cd} = 16 \cdot x$ is a square only for $x = 4$. Thus we find one solution $804^2 = 646416$.

$\overline{ab} = 81$ We write $n = 900 + x$, where x is a positive integer. We have $K = (900 + x)^2 = 810000 + 1800x + x^2$. Since $K < 820000$, it follows that $1800x < 10000$ and hence $x \leq 5$. Again we also have $x \geq 4$.

For the remaining candidates $x = 4, 5$, the number $\overline{cd} = 18 \cdot x$ is not a square. Hence, there are no solutions.

We conclude that, in total, there are two six-digit multi-squares that are themselves a square: 166464 and 646416.