Second round
Dutch Mathematical Olympiad

Friday 17 March 2017

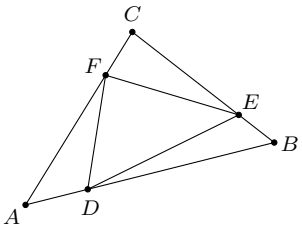
- Time available: 2.5 hours.
- The competition consists of five B-problems and two C-problems.
- Formula sheets and calculators are not allowed. You can only use a pen, compass, ruler, set square, and of course your mental skills.
- Good luck!

B-problems

For the B-problems only the answer has to be handed in (for example, a number). No explanation is required. A correct answer is awarded 4 points, for a wrong or incomplete answer no points are given. Please work very accurately: a minor error in a calculation may result in a wrong answer.
NOTE: All answers should be given in exact form, like $\frac{11}{81}$, $2+\frac{1}{2}\sqrt{5}$, $\frac{1}{4}\pi+1$, or 3^{100}.

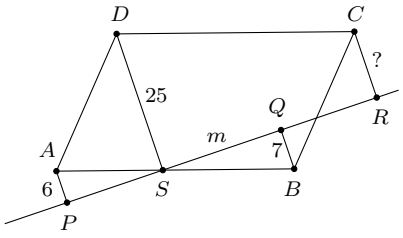
B1. A finite sequence of consecutive positive integers is called *balanced* if it contains equally many multiples of three and multiples of five. An example of a sequence of length 7 that is not balanced is 30, 31, 32, 33, 34, 35, 36, because this sequence contains 3 multiples of three (namely 30, 33, and 36) and just 2 multiples of five (namely 30 and 35).
What is the maximal length of a balanced sequence of consecutive positive integers?

B2. The area of a given triangle ABC equals 40. Point D on side AB satisfies $|BD| = 3 \cdot |AD|$. Point E on side BC satisfies $|CE| = 3 \cdot |BE|$. Point F on side CA satisfies $|AF| = 3 \cdot |CF|$.
Determine the area of triangle DEF.



B3. In math class, a student has written down a sequence of 16 numbers on the blackboard. Below each number, a second student writes down how many times that number occurs in the sequence. This results in a second sequence of 16 numbers. Below each number of the second sequence, a third student writes down how many times that number occurs in the second sequence. This results in a third sequence of numbers. In the same way, a fourth, fifth, sixth, and seventh student each construct a sequence from the previous one. Afterwards, it turns out that the first six sequences are all different. The seventh sequence, however, turns out to be equal to the sixth sequence.
Give one sequence that could have been the sequence written down by the first student.

B4. A parallelogram $ABCD$ is intersected by a line m. From each of the four vertices A, B, C, and D we draw a perpendicular to m. The four feet are P, Q, R, and S, respectively. Point S is also the intersection of line m and AB. The lengths of line segments AP, BQ, and DS are 6, 7, and 25, respectively.
What is the length of CR?
Be careful: the figure is not drawn to scale.



PLEASE CONTINUE ON THE OTHER SIDE

B5. Simon has 2017 blue blocks that are numbered from 1 up to and including 2017. He also has 2017 yellow blocks that are numbered from 1 up to and including 2017. Simon wants to arrange his 4034 blocks in a row, in such a way that, for every $k = 1, 2, \ldots, 2017$, the following conditions are met:

- to the left of blue block number k there are k or more yellow blocks;
- to the right of yellow block number k there are k or fewer blue blocks.

Determine all possible numbers for the 1000th block from the left in the row.

C-problems

For the C-problems not only the answer is important; you also have to write down a clear reasoning. Use separate sheets of paper for each C-problem. A correct and well-explained answer is awarded 10 points.
Partial solutions may also be worth some points. Therefore, write neatly and hand in your drafts (for each problem separately).

C1. You have 1000 tiles of each of the following five types:

A	B	C	D	E
1 0 1 0 1 0	1 1 0 1 1	1 1 0 1	1 0 0 0 1 0	0 1 0 0 1

You want to form a row of tiles such that the same sequence of zeroes and ones is formed on the top and the bottom. We will call this a *matching combination*. Consider, for example, the row 'DDC' consisting of three tiles of types D, D, and C, in that order. The top sequence is 1001001, while the bottom sequence is 010010101. Since the two sequences are not the same, the row of tiles is not a matching combination.

(a) Construct a matching combination using only tiles of type A, B, and C.

(b) Show that no matching combination using only tiles of types B, C, and D exists.

(c) Does a matching combination using only tiles of types B, C, D, and E exist?
If so, give an example. If not, prove that such a combination does not exist.

C2. A *multi-square* is a number obtained by concatenating two or more square two-digit numbers. (A two-digit number is not allowed to start with digit 0). For example, since 16 and 25 are squares, 1625 is a multi-square.

(a) Determine all four-digit multi-squares whose first and last digit are equal.

(b) Determine all six-digit multi-squares that are themselves squares.