57th Dutch Mathematical Olympiad 2018

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2019 started with the first round in January 2018, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In this first round 8924 students from 339 secondary schools participated.

The 1014 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 113 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 57th edition 2018.

The 30 most outstanding candidates of the Dutch Mathematical Olympiad 2018 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

In February a team of four girls was chosen from the training group to represent the Netherlands at the EGMO in Kyiv, Ukraine, from 7 until 13 April. The team brought home a silver medal, two bronze medals, and a honourable mention; a very nice achievement. For more information about the EGMO (including the 2019 paper), see www.egmo.org.

In March a selection test of three and a half hours was held to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Valkenswaard, Netherlands, from 26 until 28 April. The Dutch team received three silver medals, four bronze medals and a honourable mention. For more information about the BxMO (including the 2019 paper), see www.bxmo.org.

In May the team for the International Mathematical Olympiad 2019 was selected by three team selection tests on 29, 30 and 31 May, each lasting four hours. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Bristol, United Kingdom, from 6 until 14 July.

For younger students the Junior Mathematical Olympiad was held in October 2018 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2019 in the United Kingdom consists of

- Szabi Buzogány (19 years old)
  - silver medal at BxMO 2018, silver medal at BxMO 2019
  - honourable mention at IMO 2018
- Jesse Fitié (17 years old)
- Jovan Gerbscheid (16 years old)
  - silver medal at BxMO 2018
  - bronze medal at IMO 2018
- Jippe Hoogeveen (16 years old)
  - bronze medal at IMO 2018
- Matthijs van der Poel (18 years old)
  - bronze medal at BxMO 2016, bronze medal at BxMO 2017
  - observer C at IMO 2016, silver medal at IMO 2017, silver medal at IMO 2018
- Richard Wols (16 years old)
  - bronze medal at BxMO 2018, silver medal at BxMO 2019
  - observer C at IMO 2018

We bring as observer C the promising young student

- Tjeerd Morsch (17 years old)
  - bronze medal at BxMO 2019

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Jeroen Huijben (observer B), University of Amsterdam

First Round, January 2018

Problems

A-problems

1. In a classroom there are chairs and stools. On each chair and on each stool one child is seated. Each chair has 4 legs, each stool has 3 legs and each child has 2 legs. Together, we have a total of 39 legs.
How many chairs are there in the classroom?

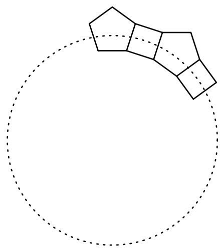
A) 3 B) 4 C) 5 D) 6 E) 9

2. On an island, there are knights and knaves. Knights always speak the truth and knaves always lie. On the island you meet five people. You know that four of them are knights and one of them is a knave, but you do not know who is the knave. They make the following statements about the island inhabitants:

- A: "All knaves have shoe size 40."
- B: "All people with shoe size 40 have a goldfish."
- C: "All people with a goldfish are knaves."
- D: "I have shoe size 40."
- E: "I have a goldfish."

Which of them is the knave?

A) A B) B C) C D) D E) E



3. If you continue the chain of squares and regular pentagons in the same way, does it connect to itself after going around? If so, how many pentagons do you need?

A) 9 B) 10 C) 11 D) 12 E) It does not connect.

4. Julian wants to compose a list of integers. He wants the list to be as long as possible. Each integer on the list must consist of one or more of the digits 1 to 9. Moreover,

- each of the digits 1 to 9 is used exactly once;
- no integer in the list is divisble by another integer in the list.

What is the maximum number of integers in Julian's list?

A) 4 B) 5 C) 6 D) 7 E) 8

5. Nine people are at a party. While entering, some of them shook hands. Quintijn is at the party and asks each of the others how many hands they shook. He gets eight different answers.
How many hands did Quintijn shake?

A) 0 B) 1 C) 2 D) 3 E) 4

6. Birgit is studying positive integers n for which n is divisible by 4, $n+1$ is divisible by 5, and $n+2$ is divisible by 6.
How many of such integers n are smaller than 2018?

A) 16 B) 17 C) 18 D) 33 E) 34

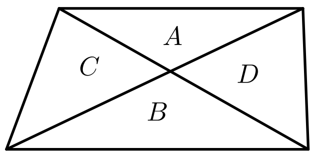
7. A frog starts in the point at coordinates $(0,0)$ in the plane. He can make three kinds of jumps:

- from (x,y) to $(x,y-5)$;
- from (x,y) to $(x-2,y+3)$;
- from (x,y) to $(x+4,y+9)$.

Ahead, there are three juicy snacks that the frog would like to eat: a worm at $(2013,2018)$, a beetle at $(2018,2019)$, and a snail at $(2018,2023)$.
Which of these snacks can the frog reach?

A) worm and snail B) beetle and snail C) worm and beetle
D) only the beetle E) only the snail

8. Harold draws a trapezium with parallel top and bottom sides. The length of the top side is smaller than the length of the bottom side. The two diagonals divide the trapezium into four triangles. The area of the upper triangle is called A, of the lower B, of the left C, and of the right D. An example of such a trapezium is depicted on the right.



Which of the following equalities holds for any such trapezium?

A) $A+C=B+D$ B) $A+D=B+C$ C) $A+B=C+D$
D) $A:B=D:C$ E) $A:C=D:B$

B-problems

The answer to each B-problem is a number.

1. Three years ago, Rosa's mother was exactly five times as old as Rosa was at that time. At that moment, Rosa's mother was just as old as Rosa's grandmother was when Rosa's mother was born. Now, Rosa's grandmother is exactly seven times as old as Rosa is.
How old is Rosa's mother now?

2. Nanda and Mike both have a note containing the same five-digit number. Nanda puts the digit 4 in front and the digit 8 at the end of her number to obtain a seven-digit number. Mike puts one digit in front of his number. Comparing their new numbers, Nanda's number turns out to be exactly 6 times as large as Mike's.
What was their starting number?

3. We consider a square, the circle through the vertices of the square and the circle touching the four sides of the square (see the left-hand figure). The ring-shaped area between the two circles is divided into four dark pieces (inside the square) and four light pieces (outside the square). The area of the square is 60.

What is the total area of two dark pieces and one light piece together as depicted in the right-hand figure?

4. Elisa is making so-called *dubious dice*. Each face of a dubious die contains one of the numbers 1 to 6, but not all these numbers need to occur and some may occur more than once. However, from every direction it must look like a real die. This means that in each corner three different numbers must meet, no two of which add up to 7 (on a real die such pairs are always on opposite faces). For example, the numbers 1, 2, and 4 may meet in a corner, but 1, 2, and 5 may not as $2 + 5 = 7$. Of course, a normal die is an example of a dubious die as well.

Elisa is interested in the sum of the six numbers on a dubious die.
How many possible values are there for this sum?

Solutions

A-problems

1. B) 4

2. C) C

3. B) 10

4. D) 7

5. E) 4

6. E) 34

7. E) only the snail

8. E) $A : C = D : B$

B-problems

1. 33 years

2. 49998

3. 15

4. 19

Second Round, March 2018

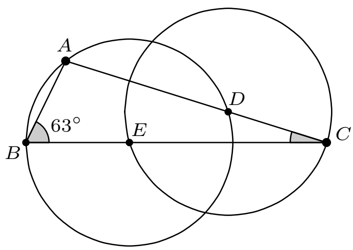
Problems

B-problems

The answer to each B-problem is a number.

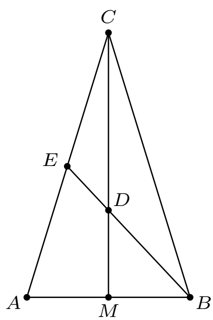
B1. Anouk, Bart, Celine, and Daan have participated in a math competition. Each of their scores is a positive integer. The sum of the scores of Bart and Daan is the same as the sum of the scores of Anouk and Celine. The sum of the scores of Anouk and Bart is higher than the sum of the scores of Celine and Daan. Daan's score is higher than the sum of the scores of Bart and Celine.
Write down the names of the four students in decreasing order of their scores.

B2. In the figure, you can see a triangle ABC. The angle at B is equal to 63 degrees. Side AC contains a point D, and side BC contains a point E. Points B, A, and D lie on a circle with centre E. Points E and C lie on a circle with centre D.
What is the angle at point C?



B3. A *palindromic number* is a positive integer (consisting of one or more digits) that remains the same when the digits are reversed. For example: 1245421 and 333 are palindromic numbers, but 345 and 100 are not. There is exactly one palindromic number n with the following property: if you subtract 2018 from n, the result is again a palindromic number.
What number is n?

B4. Triangle ABC is isosceles with apex C. The midpoint of AB is point M. On segment CM there is a point D such that $\frac{|CD|}{|DM|} = \frac{3}{2}$. Line BD intersects segment AC in point E.
Determine $\frac{|CE|}{|EA|}$.



B5. A *sawtooth number* is a positive integer with the following property: for any three adjacent digits, the one in the middle is either greater than its two neighbours or smaller than its two neighbours. For example, the numbers 352723 and 314 are sawtooth numbers, but 3422 and 1243 are not.
How many 8-digit sawtooth numbers exist, for which each of the digits is equal to 1, 2, or 3?

C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. You have n balls that are numbered from 1 to n. You need to distribute the balls over two boxes. The *value* of a box is the sum of the numbers of the balls in that box. Your distribution of balls must obey the following rules:

- Each box has at least one ball.
- The two boxes do not have the same number of balls.
- The value of the box with the least number of balls must be at least 2 more than the value of the box with the most balls.

Determine for which positive integers n this is possible.
(Prove that for those values of n it is indeed possible, and prove that it is not possible for other values of n.)

C2. In this problem we consider squares: numbers of the form m^2 where m is an integer.

(a) Does there exist an integer a such that $16 + a$, $3 + a$, and $16 \cdot 3 + a$ are squares?
If so, give such a number a and show that the three numbers are indeed squares.
If not, prove that such a number a does not exist.

(b) Does there exist an integer a such that $20 + a$, $18 + a$, and $20 \cdot 18 + a$ are squares?
If so, give such a number a and show that the three numbers are indeed squares.
If not, prove that such a number a does not exist.

(c) Prove that for every odd integer n there exists an integer a such that $2018 + a$, $n + a$, and $2018 \cdot n + a$ are squares.

Solutions

B-problems

1. A, D, B, C

2. 18 degrees

3. 2442

4. $\frac{3}{4}$

5. 110

C-problems

C1. A distribution of the balls that follows all three rules will be called a *correct* distribution. The box containing the larger number of balls will be called the *fullest* box. It is clear that at least $1 + 2 = 3$ balls are needed to obey the first two rules.

We first consider the case that n is **odd**.
If $n = 3$, then the fullest box must have two balls. It therefore has a value of at least $1 + 2 = 3$, while the other box has one ball and hence a value of at most 3. The third rule is broken so there is no correct distribution.

For $n = 5$ there is a correct distribution: put balls 1, 2, and 3 in one box and put balls 4 and 5 in the other box. Since $4 + 5$ is at least two more than $1 + 2 + 3$, this is indeed a correct distribution.

If there is a correct distribution for n balls, there is also one for $n + 2$ balls. Indeed, we can simply add ball $n + 1$ to the fullest box and add ball $n + 2$ to the other box. The value of the fullest box increases by less than the other box, hence we still follow rule 3.

Since we have a correct distribution for $n = 5$, we also have a correct distribution for $n = 7$. Then, we also find a correct distribution for $n = 9$, $n = 11$, et cetera.

Now we consider the case that n is **even**.
If $n = 4$, the fullest box must have at least three balls and hence has a value of at least $1 + 2 + 3 = 6$. The other box has only one ball and has a

value of at most 4. This means that the third rule is broken. There is no correct distribution.

If $n = 6$, the fullest box must have at least four balls. It therefore has a value of at least $1 + 2 + 3 + 4 = 10$. The other box has at most two balls and therefore has a value of at most $5 + 6 = 11$. Since $11 < 2 + 10$, there is no correct distribution.

For $n = 8$, there is a correct distribution: put balls 1 to 5 into one box and put balls 6 to 8 in the other box. The fullest box then has a value of $1 + 2 + 3 + 4 + 5 = 15$, and the other box has a value of $6 + 7 + 8 = 21$. Since $21 \geq 2 + 15$, this is indeed a correct distribution.

Again, we see that a correct distribution of n balls gives a correct distribution with $n + 2$ balls by adding ball $n + 1$ to the fullest box and adding ball $n + 2$ to the other box. Since we have a correct distribution for $n = 8$, we thus find correct distributions for $n = 10, 12, 14, \ldots$.

We conclude: for $n = 1, 2, 3, 4, 6$ there is no correct distribution, but for all other positive integers n a correct distribution does exist.

C2. (a) Yes, such a number a exists. For example, take $a = 33$. Then we have: $16 + a = 7^2$, $3 + a = 6^2$, and $16 \cdot 3 + a = 9^2$.
A suitable a can be easily found by trying $48 + a = 49$, 64, 81.

(b) No, such a number a does not exist. The numbers $20 + a$ and $18 + a$ cannot both be squares since the difference of two squares is never equal to 2. Indeed, suppose that $m^2 - n^2 = 2$. Then we would have $(m+n)(m-n) = 2$, where $m > n$. Since 1 and 2 are the only positive divisors of 2, we would have $m+n = 2$ and $m-n = 1$. But this would imply that $2m = (m + n) + (m - n) = 3$, which is impossible since $2m$ is even whereas 3 is odd.

(c) Given an odd integer n, we will show that there exist integers a, x, y, and z such that

$$\begin{aligned} 2018 + a &= x^2, & (1)\\ n + a &= y^2, & (2)\\ 2018n + a &= z^2. & (3) \end{aligned}$$

If we subtract equation (2) from equation (1), we obtain

$$2018 - n = x^2 - y^2 = (x - y)(x + y).$$

We therefore choose x and y such that $x+y = 2018-n$ and $x-y=1$. Addition of these two equations gives us $2x = 2019-n$, subtraction gives us $2y = 2017-n$. We therefore choose

$$x = \frac{2019-n}{2} \qquad \text{and} \qquad y = \frac{2017-n}{2}.$$

Since n is odd, these are integers. Indeed, we now have $x-y=1$ and $x+y=2018-n$, and hence that

$$2018 - n = x^2 - y^2. \tag{4}$$

We let $a = y^2 - n$. Then equation (2) certainly holds. Equation (1) holds as well, since $2018+a = 2018+y^2-n = x^2$ (the second equation follows from (4)).

Finally, we show that $2018n+a$ is a square. We successively obtain

$$\begin{aligned}
2018n + a &= 2018n + y^2 - n \\
&= 2017n + \frac{(2017-n)^2}{4} \\
&= \frac{4 \cdot 2017n + 2017^2 - 2 \cdot 2017n + n^2}{4} \\
&= \frac{2017^2 + 2 \cdot 2017n + n^2}{4} \\
&= \frac{(2017+n)^2}{4} = \left(\frac{2017+n}{2}\right)^2.
\end{aligned}$$

Since n is odd, the number $\frac{2017+n}{2}$ is an integer. By choosing $z = \frac{2017+n}{2}$, we have found integers a, x, y, and z for which equations (1), (2), and (3) are true.

Final Round, September 2018

Problems

1. We call a positive integer a *shuffle number* if the following hold:

(1) All digits are nonzero.

(2) The number is divisible by 11.

(3) The number is divisible by 12. If you put the digits in any other order, you again have a number that is divisible by 12.

How many 10-digit shuffle numbers are there?

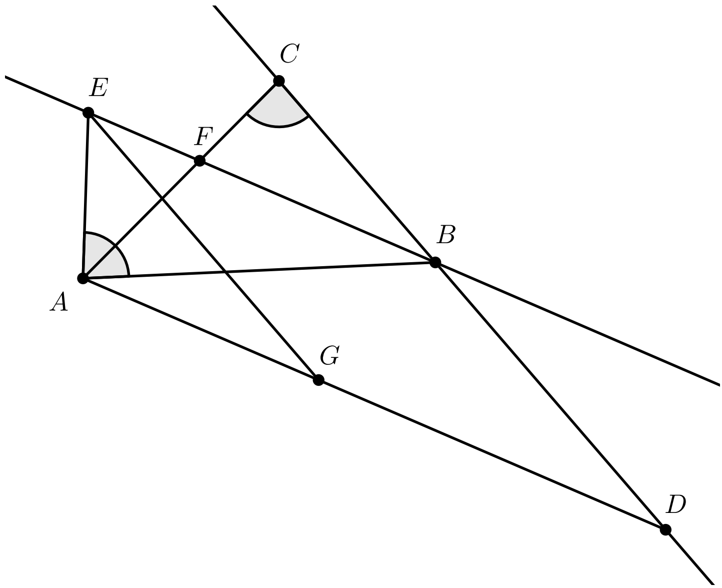
2. The numbers 1 to 15 are each coloured blue or red. Determine all possible colourings that satisfy the following rules:

- The number 15 is red.
- If numbers x and y have different colours and $x + y \leq 15$, then $x + y$ is blue.
- If numbers x and y have different colours and $x \cdot y \leq 15$, then $x \cdot y$ is red.

3. Determine all triples (x, y, z) consisting of three *distinct* real numbers, that satisfy the following system of equations:

$$\begin{aligned} x^2 + y^2 &= -x + 3y + z, \\ y^2 + z^2 &= x + 3y - z, \\ x^2 + z^2 &= 2x + 2y - z. \end{aligned}$$

4. In triangle ABC, $\angle A$ is smaller than $\angle C$. Point D lies on the (extended) line BC (with B between C and D) such that $|BD| = |AB|$. Point E lies on the bisector of $\angle ABC$ such that $\angle BAE = \angle ACB$. Line segment BE intersects line segment AC in point F. Point G lies on line segment AD such that EG and BC are parallel.
Prove that $|AG| = |BF|$.



5. At a quiz show there are three doors. Behind exactly one of the doors, a prize is hidden. You may ask the quizmaster whether the prize is behind the left-hand door. You may also ask whether the prize is behind the right-hand door. You may ask each of these two questions multiple times, in any order that you like. Each time, the quizmaster will answer 'yes' or 'no'. The quizmaster is allowed to lie at most 10 times. You have to announce in advance how many questions you will be asking (but which questions you will ask may depend on the answers of the quizmaster).
What is the smallest number you can announce, such that you can still determine with absolute certainty the door behind which the prize is hidden?

Solutions

1. First observe that a shuffle number can only contain the digits 2, 4, 6, and 8. Indeed, if we place the digits in any order, we obtain an *even* number (since it is divisible by 12) because of property (3). Since the last digit of an even number is also even, the digit must be 2, 4, 6, or 8 since it cannot be 0 because of property (1). Since we can put any of the digits in the last position, this holds for each digit of a shuffle number.

Next, observe that a shuffle number can only contain the digits 4 and 8. Indeed, suppose that we had a shuffle number containing the digit 2. We could reorder the digits so that the last digit is 2. The last two digits would then be 22, 42, 62, or 82. But then the number would not be divisible by 4 (and hence also not divisible by 12), contradicting property (3). In the same way we see that a shuffle number cannot contain digit 6 because a number ending in digits 26, 46, 66, or 86 is not divisible by 4.

A shuffle number is divisible by 3 (since it is divisible by 12), hence the sum of its digits is divisible by 3 as well. If our 10-digit number has k eights and $10-k$ fours, then the sum of its digits is equal to $8k+4(10-k)=40+4k=36+4(k+1)$. This is divisible by 3 if and only if $k+1$ is divisible by 3. That is, if $k=2$, $k=5$, or $k=8$. We see that a shuffle number must have 2 eights and 8 fours, or 5 eights and 5 fours, or 8 eights and 2 fours. Note that each of those numbers satisfy (1) and (3). It remains to examine which of these numbers are divisible by 11 (prop. (2)).

For this we use the 11-criterion: a number is divisible by 11 if the *alternating sum* of the digits is divisible by 11. By alternating sum we mean that instead of adding them, we alternately add and subtract. Since in our case all digits are equal to 4 or to 8, the alternating sum must even be divisible by $4\times 11=44$. Since the alternating sum cannot be greater than $8+8+8+8+8-4-4-4-4-4=20$ (and cannot be smaller than -20), it must be equal to 0. In other words: the sum of the five digits in the odd positions must equal the sum of the five digits in the even positions. This means that the number of digits 8 in the odd positions must be equal to the number of digits 8 in the even positions. We examine the three cases that we found before:

- Suppose exactly 2 digits are eights. The only requirement is now that there is exactly 1 eight in the odd positions and exactly 1 eight in the even positions. There are $5\times 5=25$ ways to do this.
- Suppose exactly 5 digits are eights. Since we cannot divide the eights into two equal groups, there are no solutions in this case.

- Suppose exactly 8 digits are eights. The only requirement is now that we have 4 eights in the odd positions and 4 eights in the even positions. In other words: 1 odd position must be a four and 1 even position must be a four. Again we find $5 \times 5 = 25$ possibilities.

We conclude that the total number of 10-digit shuffle numbers is $25+25 = 50$.

2. We first consider the case that 1 is red. Then all numbers from 1 to 15 are red. Indeed, suppose that some number k is blue. Then 1 and k have different colours, hence by the third rule the number $1 \cdot k = k$ must be red. But this contradicts the assumption that it was blue. Observe that colouring all numbers red indeed gives a correct colouring.

Now we consider the case that 1 is blue. Observe that when two numbers sum to 15, those numbers must have the same colour by the second rule. We get the following pairs of numbers of the same colour: 1 and 14, 2 and 13, 3 and 12, 4 and 11, 5 and 10, 6 and 9, 7 and 8.

The number 2 is blue. Indeed, suppose that 2 is red. From the second rule, we derive that $3 = 1 + 2$ is blue. Repeatedly applying the same rule we find that $5 = 3 + 2$ is blue, that $7 = 5 + 2$ is blue, and finally that 15 is blue. Since 15 is in fact not blue, 2 cannot be red.

The number 7 is blue. Indeed, suppose that 7 is red. It then follows from the second rule that $8 = 1 + 7$ is blue. However, 7 and 8 must have the same colour, so this cannot be the case.

The number 4 is blue. Indeed, suppose that 4 is red. It then follows from the second rule that $11 = 4 + 7$ is blue. But 4 and 11 have the same colour, so this cannot be the case.

Recall, that 3 and 12 have the same colour, and so do 6 and 9. In fact, all four numbers must have the same colour. Indeed, otherwise $9 = 3 + 6$ would be blue by the second rule and also $12 = 3 + 9$ would be blue by the second rule.

So far, we know that 15 is red, that $1, 2, 4, 7, 8, 11, 13, 14$ are blue, that $3, 6, 9, 12$ have the same colour, and that $5, 10$ have the same colour. The numbers 3 and 5 cannot both be red, since if 3 is red, $5 = 2 + 3$ is blue by the second rule. The three remaining colour combinations for the numbers 3 and 5 result in the following three colourings:

(1) Only 15 is red.

(2) Only $5, 10, 15$ (the numbers divisible by 5) are red.

(3) Only $3, 6, 9, 12, 15$ (the numbers divisible by 3) are red.

It is easy to check that all three colourings are indeed correct. We write this out for the third colouring, the other two can be checked in a similar way. That the sum of a red and a blue number is always blue follows from the fact that the sum of a number divisible by 3 and a number not divisible by 3 is itself not divisible by 3. That the product of a blue and a red number is always red follows from the fact that the product of a number divisible by 3 and a number not divisible by 3 is itself divisible by 3.

We conclude that there are 4 correct colourings in total: the colouring in which all numbers are red, and colourings (1), (2), and (3).

3. If we subtract the second equation from the first, we obtain

$$x^2 - z^2 = -x + z - (x - z),$$

which can be rewritten as

$$(x - z)(x + z) = -2(x - z),$$

and then simplified further to get

$$(x - z)(x + z + 2) = 0.$$

Since x and z must be different, $x - z$ is nonzero and we can conclude that $x + z + 2 = 0$, hence $z = -x - 2$. If we subtract the third equation from the second, we obtain

$$y^2 - x^2 = x + 3y - (2x + 2y),$$

which can be rewritten as

$$(y - x)(y + x) = 1(y - x),$$

and then simplified further to get

$$(y - x)(y + x - 1) = 0.$$

Since x and y must be different, $y - x$ is nonzero and we can conclude that $y + x - 1 = 0$, hence $y = 1 - x$.

If we now substitute $y = 1 - x$ and $z = -x - 2$ in the first equation, we get

$$x^2 + (1 - x)^2 = -x + 3(1 - x) + (-x - 2),$$

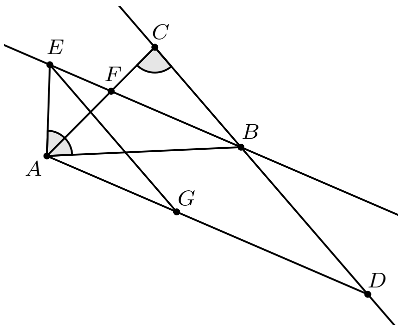
which we can rewrite as

$$2x^2 - 2x + 1 = -5x + 1.$$

This is a quadratic equation $2x^2 + 3x = 0$, whose roots are $x = 0$ and $x = -\frac{3}{2}$. In both cases we can deduce the values of y and z from the formulas $y = 1 - x$ and $z = -x - 2$. This gives the solutions $(x, y, z) = (-\frac{3}{2}, \frac{5}{2}, -\frac{1}{2})$ and $(x, y, z) = (0, 1, -2)$.

For these solutions we know that they satisfy the first equation. Since the difference between the first and the second equation gives zero on both sides (since we chose $z = -x - 2$), the second equation is satisfied as well. Because of the choice $y = 1 - x$, also the third equation is satisfied. We conclude that both solutions indeed satisfy all three equations.

4. Because BE is the bisector of $\angle ABC$, it follows that $\angle ABE = \angle CBF$. It was given that $\angle FCB = \angle EAB$. Since the angles in a triangle sum to $180°$, both in triangle ABE and in triangle CBF, we must also have $\angle BFC = \angle BEA$. Since $\angle BFC$ and $\angle AFE$ are opposite angles, we see that $\angle AFE = \angle FEA$, and hence that triangle AEF is isosceles.



We will show that $\triangle ABF$ and $\triangle EGA$ are congruent.

We see that $\angle EGA = \angle CDA$, since EG and BC are parallel. In the isosceles triangle ABD, we see that $\angle CDA = \frac{1}{2}(180° - \angle ABD)$, which is equal to $\frac{1}{2}\angle ABC$ (because of the straight angle). This in turn, is equal to $\angle ABF$, because BF is the bisector of $\angle ABC$. Altogether, we find that $\angle EGA = \angle ABF$.

Triangle ABD is isosceles, so $\angle BAG = \angle CDA$. As we just observed, this last angle is equal to $\angle ABF$. We have also seen that $\angle EGA = \angle ABF$. Because the three angles in triangle GEA sum to $180°$, we get

$$\angle GEA = 180° - \angle EGA - \angle EAB - \angle BAG = 180° - \angle ABF - \angle FCB - \angle ABF.$$

Since $2 \cdot \angle ABF = \angle ABC$ (since BF is the bisector) we thus find that $\angle GEA = 180° - \angle FCB - \angle ABC = \angle BAC$ (because the angles in triangle ABC sum to $180°$), hence $\angle GEA = \angle BAF$.

Using the fact that $|AE| = |AF|$, which follows from the fact that $\triangle AEF$ is isosceles, we can now conclude that $\triangle ABF$ and $\triangle EGA$ are congruent. This immediately implies $|AG| = |BF|$ as required.

5. The smallest number of questions that suffices is 32. First we will show a strategy to locate the prize in only 32 questions.

Start by asking the quizmaster if the prize is behind the left-hand door. Repeat this until you are sure whether the prize is there or not. You can only be 100% sure when you have received the same answer 11 times, because the quizmaster can lie a maximum of 10 times. After doing this, suppose that the quizmaster has lied n times. This means that you have thus far asked $11 + n$ questions and the quizmaster is entitled to $10 - n$ more lies.

Now ask the quizmaster whether the prize is behind the right-hand door, and keep asking this until you are 100% sure of the true answer. Since the quizmaster can lie only $10-n$ more times, you need at most $2(10-n)+1 = 20-2n+1$ questions for this. In total you have now asked $32-n$ questions. We conclude that with this strategy, 32 questions always suffice.

Having only 31 questions, it is not always possible to locate the prize. We will show how the quizmaster can make sure of that. At the beginning, as long as you have asked about both doors at most 10 times, he always answers 'no'. To keep it simple, we assume that the left-hand door is the first door that you query for the eleventh time (the other case is completely analogous). We consider the situation that the prize is not behind the left-hand door (which may happen). In this case, we show that the quizmaster can make sure that after 31 questions, you cannot tell whether the prize is behind the middle door or behind the right-hand door.

So far, the quizmaster has been speaking the truth about the left-hand door, and he will continue to do so: when asked about the left-hand door, he will always answer 'no'. For the right-hand door, the quizmaster will keep saying 'no' up to and including the tenth time he is asked about this door. The next ten times he is asked about this door, he will answer 'yes'. Since you ask at most 20 questions about the right-hand door (since you already queried the left-hand door at least 11 times), the quizmaster needs to lie at most 10 times. Whether the prize is behind the middle door or behind the right-hand door, in both cases the quizmaster gives the same 31 answers. Therefore, you cannot determine the door behind which the prize is located.

BxMO Team Selection Test, March 2019

Problems

1. Prove that for each positive integer n there are at most two pairs (a, b) of positive integers with following two properties:

(i) $a^2 + b = n$,

(ii) $a+b$ is a power of two, i.e. there is an integer $k \geq 0$ such that $a+b = 2^k$.

2. Let ABC be a triangle and let I be the incentre of this triangle. The line through I perpendicular to AI intersects the circumcircle of $\triangle ABC$ in points P and Q, where P lies on the same side of AI as B. Let S be the second intersection point of the circumcircles of $\triangle BIP$ and $\triangle CIQ$. Prove that SI is the angle bisector of $\angle PSQ$.

3. Let x and y be positive real numbers.

1. Prove: if $x^3 - y^3 \geq 4x$, then $x^2 > 2y$.
2. Prove: if $x^5 - y^3 \geq 2x$, then $x^3 \geq 2y$.

4. Do there exist a positive integer k and a non-constant sequence $a_1, a_2, a_3, \ldots$ of positive integers such that $a_n = \gcd(a_{n+k}, a_{n+k+1})$ for all positive integers n?

5. In a country, there are 2018 cities, some of which are connected by roads. Each city is connected to at least three other cities. It is possible to travel from any city to any other city using one or more roads. For each pair of cities, consider the shortest route between these two cities. What is the greatest number of roads that can be on such a shortest route?

Solutions

1. Suppose there are three or more of such pairs for one particular n. Then by the pigeonhole principle, there are at least two such pairs, say (a, b) and (c, d), with $a \equiv c \mod 2$. Write $a + b = 2^k$ and $c + d = 2^\ell$ where k and ℓ are positive integers. Without loss of generality, assume that $\ell \geq k$. We see that

$$\begin{aligned} 2^\ell - 2^k &= (c + d) - (a + b) \\ &= (c + n - c^2) - (a + n - a^2) = c - a - c^2 + a^2 \\ &= (a + c - 1)(a - c). \end{aligned}$$

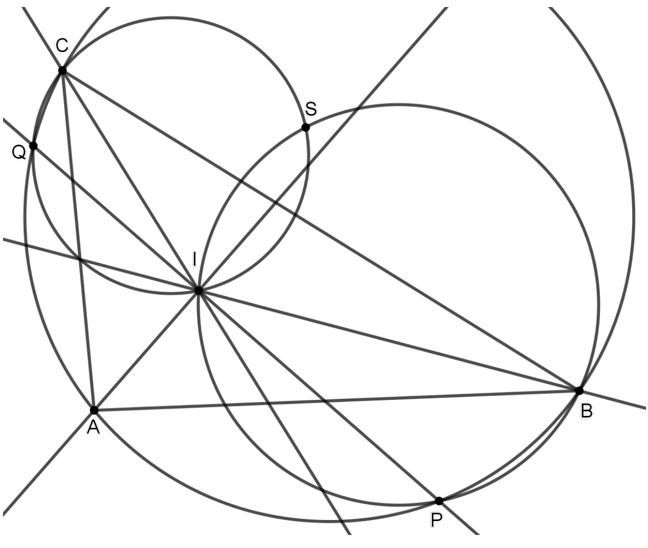
As $\ell \geq k$, we have that 2^k is a divisor of $2^\ell - 2^k$. Because $a \equiv c \mod 2$, the integer $a + c - 1$ is odd, hence 2^k is a divisor of $a - c$. As $2^\ell - 2^k \geq 0$ and $a + c - 1 > 0$, we have $a - c \geq 0$. On the other hand, we have $a - c < a + b = 2^k$, hence $0 \leq a - c < 2^k$, while $2^k \mid a - c$. We conclude that $a - c = 0$, hence $a = c$. Now we have $b = n - a^2 = n - c^2 = d$, hence the two pairs (a, b) and (c, d) are identical, which gives a contradiction. Therefore, there at most two pairs (a, b) having the two properties. $\square$

2. Write $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$, and $\angle BCA = 2\gamma$. As the angles in triangle $\triangle AIB$ add up to $180°$, we have

$$\angle BIA = 180° - \angle IAB - \angle ABI = 180° - \alpha - \beta = 90° + \gamma,$$

hence

$$\angle BIP = \angle BIA - \angle AIP = 90° + \gamma - 90° = \gamma = \angle BCI.$$



Moreover, as $BPQC$ is a cyclic quadrilateral, we have

$$180^\circ - \angle BPI = 180^\circ - \angle BPQ = \angle BCQ = \angle BCI + \angle ICQ.$$

If we now consider the sum of the angles in triangle BPI, and use both previous results, we find

$$\angle PBI = 180^\circ - \angle BPI - \angle BIP = \angle BCI + \angle ICQ - \angle BCI = \angle ICQ.$$

Using the fact that $BSIP$ and $CQIS$ are cyclic quadrilaterals, we get

$$\angle PSI = \angle PBI = \angle ICQ = \angle ISQ,$$

hence SI is the angle bisector of angle PSQ. $\square$

3.
1. We have $x^3 - 4x \geq y^3 > 0$, hence $x(x^2 - 4) > 0$. As x is positive, this yields $x^2 - 4 > 0$, hence $x^2 > 4$. That means that $x > 2$. Moreover, we have $x^3 - y^3 \geq 4x > 0$, hence $x > y$. The combination of these two results (which is allowed because x and y are both positive) gives $x^2 = x \cdot x > 2 \cdot y = 2y$.
2. We have $(x^4 - 4)^2 \geq 0$. Expanding yields $x^8 - 8x^4 + 16 \geq 0$. Because x is positive, we can multiply this with x without changing the inequality sign, hence we have $x^9 \geq 8x^5 - 16x$. The inequality in the assumption gives $x^5 - 2x \geq y^3$. If we combine this with the preceding inequality, we get $x^9 \geq 8(x^5 - 2x) \geq 8y^3$. It follows that $x^3 \geq 2y$. $\square$

4. Such a k and a sequence do not exist. We prove this by contradiction, so suppose they do exist. Note that $a_n \mid a_{n+k}$ and $a_n \mid a_{n+k+1}$ for all $n \geq 1$. Using simple induction, it follows that $a_n \mid a_{n+\ell k}$ and $a_n \mid a_{n+\ell k+\ell}$ for all $\ell \geq 0$. We will prove by induction to m that $a_n \mid a_{n+mk+(m+1)}$ for all $0 \leq m \leq k-1$. For $m = k-1$, this follows from $a_n \mid a_{n+k\cdot k} = a_{n+(k-1)k+k}$. Now suppose that for a certain m with $1 \leq m \leq k-1$ we have that $a_n \mid a_{n+mk+(m+1)}$. We also know that $a_n \mid a_{n+mk+m}$. Therefore, as $a_{n+(m-1)k+m} = \gcd(a_{n+mk+m}, a_{n+mk+m+1})$, we also have $a_n \mid a_{n+(m-1)k+m}$. This finishes the induction argument. Substituting $m = 0$ yields $a_n \mid a_{n+1}$.

Because $a_n \mid a_{n+1}$, we also have $\gcd(a_n, a_{n+1}) = a_n$ for all n. Hence, $a_n = a_{n-k}$ for all $n \geq k+1$. Now we have $a_{n-k} \mid a_{n-k+1} \mid a_{n-k+2} \mid \ldots \mid a_n = a_{n-k}$. Because these are all positive integers, $a_{n-k}, a_{n-k+1}, \ldots, a_n$ must all be equal. This must be true for all $n \geq k+1$, hence the sequence is constant, which gives a contradiction. $\square$

5. The greatest number of roads that can be in such a shortest route, is 1511. We first describe a country for which this number is attained. Divide the cities in 504 groups: two groups of five cities $(A_0, B_0, C_0, D_0, E_0)$ and $(A_{503}, B_{503}, C_{503}, D_{503}, E_{503})$, and groups of four cities (A_i, B_i, C_i, D_i) for $1 \leq i \leq 502$. For all i with $0 \leq i \leq 503$: connect A_i to B_i, connect A_i to C_i, connect B_i to D_i, and connect C_i to D_i. Moreover, connect E_0 to A_0, B_0, and C_0. Connect E_{503} to B_{503}, C_{503}, and D_{503}. For each $1 \leq i \leq 502$: connect B_i to C_i. Finally, for each $0 \leq i \leq 502$: connect D_i to A_{i+1}. Now every city is connect two exactly three other cities.

If we now want to travel from A_0 to D_{503}, then we have to travel through all A_i and D_i, because the only connections between the different groups are there, and each group is only connected to the previous and the next group. Moreover, for $0 \leq i \leq 503$, the city A_i is not connected to D_i, which causes the route within group i to go through either B_i or C_i. At least three of the four or five cities of each group must lie on the route. In total, this route visits at least $3 \cdot 504 = 1512$ cities and has at least 1511 roads.

We will now prove that the shortest route between two cities cannot have more than 1511 roads. Consider such a shortest route which visits the cities A_0, A_1, ..., A_k consecutively. Each of the cities A_i has one more neighbour besides A_{i-1} and A_{i+1}, which we will call B_i (note that the cities B_0, B_1, ..., B_k do not have to be distinct). Moreover, A_0 and A_k are connected to a third city, say $C_0 \neq B_0$ and $C_k \neq B_k$ respectively. If one of the cities B_i or C_i equals one of the cities A_j, then we could have found a shorter route by going directly from A_i to A_j (or vice versa), which would be a contradiction. Hence, the cities B_i and C_i are not equal to any of the cities A_j.

If one of the cities B_i is connected to four cities A_j, say B_i is connected to A_i, A_m, A_n, and A_p with $i < m < n < p$, then we could shorten the route by going from A_i to B_i and then to A_p. This makes us skip at least two cities of the original route (A_m and A_n) and in their place we only visit B_i, hence this new route is shorter, which would be a contradiction. Therefore, within the cities B_i with $3 \leq i \leq k-3$ there are at least $\frac{k-5}{3}$ distinct cities. For B_0 and C_0, the route can only be shortened if one of these cities equals B_i for certain $i \geq 3$. That would be a contradiction, hence B_0 and C_0 are two distinct cities, not equal to B_i for $i \geq 3$. In the same way, B_k and C_k are two distinct cities, not equal to B_i for $i \leq k-3$. Altogether, we have $k+1$ cities on the route itself and at least $\frac{k-5}{3} + 2 + 2$ other cities. Therefore, $\frac{k-5}{3} + k + 5 \leq 2018$, hence $4k - 5 + 15 \leq 3 \cdot 2018 = 6054$, hence $4k \leq 6044$, hence $k \leq 1511$.

We conclude that the greatest number of roads occurring in a shortest route is 1511. $\square$

IMO Team Selection Test 1, May 2019

Problems

1. Let $P(x)$ be a quadratic polynomial with two distinct real roots. For all real numbers a and b satisfying $|a|, |b| \geq 2017$, we have $P(a^2+b^2) \geq P(2ab)$. Show that at least one of the roots of P is negative.

2. Write S_n for the set $\{1, 2, \ldots, n\}$. Determine all positive integers n for which there exist functions $f\colon S_n \to S_n$ and $g\colon S_n \to S_n$ such that for every x exactly one of the equalities $f(g(x)) = x$ and $g(f(x)) = x$ holds.

3. Let n be a positive integer. Determine the maximum value of $\gcd(a, b) + \gcd(b, c) + \gcd(c, a)$ for positive integers a, b, c such that $a + b + c = 5n$.

4. We are given a triangle ABC. On edge AC there are points D and E such that the order of points on this line is A, E, D, C. The line through E parallel to BC intersects the circumcircle of $\triangle ABD$ in a point F, with E and F lying on opposite sides of AB. The line through E parallel to AB intersects the circumcircle of $\triangle BCD$ in a point G, with E and G lying on opposite sides of BC. Prove that $DEFG$ is a cyclic quadrilateral.

Solutions

1. Write $P(x) = c(x-d)(x-e)$, where d and e are the distinct roots of P, so $d \neq e$. We also have $c \neq 0$, since otherwise P is not quadratic. We distinguish two cases. First suppose that $c > 0$. Substituting $b = -a = -2017$ in $P(a^2+b^2) \geq P(2ab)$, we obtain $P(2a^2) \geq P(-2a^2)$, and therefore

$$c(2a^2 - d)(2a^2 - e) \geq c(-2a^2 - d)(-2a^2 - 2).$$

Dividing both sides by the positive real number c, expanding, and then canceling the terms $4a^4$ and de, we get

$$-(d+e) \cdot 2a^2 \geq (d+e) \cdot 2a^2,$$

or equivalently,

$$4a^2(d+e) \leq 0.$$

Dividing by $4a^2 = 4 \cdot 2017^2$, we see that $d + e \leq 0$. As the roots d and e are distinct, one of them must be negative.

Now consider the remaining case $c < 0$. Then the graph of P opens up to the bottom, and therefore is descending on the right of the apex of the parabola. Choose any $a \neq b$ with $a, b \geq 2017$ and a, b both to the right side of the apex. Then by AM-GM we have $a^2 + b^2 > 2ab$. Since a, b are positive, at least 1, and to the right of the apex, we have $2ab > a$, so $2ab$ (and therefore $a^2 + b^2$ as well) also lies to the right of the apex. Since P is descending on the right side of the apex, we have $P(a^2 + b^2) < P(2ab)$, contrary to the given condition. We deduce that the case $c < 0$ is impossible and therefore that $c > 0$, from which we conclude that at least one of the roots of P is negative. $\square$

2. We first show that if $n = 2m$ for some positive integer m, such functions exist. Define

$$f(x) = \begin{cases} x & \text{if } 1 \leq x \leq m, \\ x - m & \text{if } m+1 \leq x \leq 2m, \end{cases} \qquad g(x) = \begin{cases} x + m & \text{if } 1 \leq x \leq m, \\ x & \text{if } m+1 \leq x \leq 2m. \end{cases}$$

First note that all values lie in S_n, so these indeed define functions $S_n \to S_n$. The range of f is equal to $\{1, 2, \ldots, m\}$ and that of g is equal to $\{m+1, m+2, \ldots, 2m\}$. So $f(g(x)) \neq x$ if $x \geq m+1$ and $g(f(x)) \neq x$ if $x \leq m$. Moreover, for $x \leq m$ we have $f(g(x)) = f(x+m) = x+m-m = x$, and for $x \geq m+1$ we have $g(f(x)) = g(x-m) = x-m+m = x$. Therefore for every $x \in S_n$, exactly one of $f(g(x)) = x$ and $g(f(x)) = x$ holds.

Now suppose that n is odd, say equal to $2m + 1$, and suppose that f and g are functions satisfying the given conditions. Without loss of generality, we assume that $f(g(x)) = x$ for (at least) $m + 1$ elements of S_n, say for $x = x_1, \ldots, x_{m+1}$. Suppose that for some i, j with $i \leq i, j \leq m + 1$ we have $g(x_i) = x_j$. Then $f(x_j) = f(g(x_i)) = x_i$, so $g(f(x_j)) = g(x_i) = x_j$. But now we have both $f(g(x)) = x$ and $g(f(x)) = x$ for $x = x_j$, which gives a contradiction. So we see that for all i with $1 \leq i \leq m + 1$ the number $g(x_i)$ is not equal to any of the x_j with $1 \leq j \leq m + 1$. Since there are m numbers in S_n not equal to any of the x_j with $1 \leq j \leq m+1$, and since there are $m + 1$ possible values of i, it follows that two of the $g(x_i)$ must be equal, say $g(x_k) = g(x_l)$ with $1 \leq k < l \leq m + 1$. But then $x_k = f(g(x_k)) = f(g(x_l)) = x_l$, which gives a contradiction. Therefore no such f and g can exist if n is odd.

Hence n satisfies the given conditions if and only of n is even. □

3. Write $G = \gcd(a, b) + \gcd(b, c) + \gcd(c, a)$. Without loss of generality we assume that $a \leq b \leq c$. Then we have $\gcd(a, b) \leq a$, $\gcd(b, c) \leq b$, and $\gcd(c, a) \leq a$. Hence

$$G \leq a + b + a \leq a + b + c \leq 5n.$$

If $3 \mid n$, then we can achieve $G = 5n$ via $a = b = c = \frac{5}{3}n$. Since all three gcd's equal $\frac{5}{3}n$, we see that $G = 3 \cdot \frac{5}{3}n = 5n$.

So suppose that $3 \nmid n$. Then a, b, and c cannot all be equal to each other. We distinguish a number of cases.

Case 1a. $b = c$ and $b \leq 2n$. Since in this case we have $a \neq b$, it follows that $\gcd(a, b) = \gcd(a, b-a) \leq b-a$. Hence $G \leq (b-a)+b+a = 2b \leq 4n$.

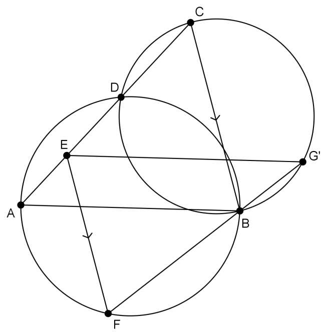
Case 1b. $b = c$ and $b > 2n$. We have $a+2b = 5n$, so $a = 5n-2b$, from which we deduce that $G \leq a+b+a = 10n-4b+b = 10n-3b < 10n-6n = 4n$.

Case 2a. $b \neq c$ and $c-a \geq n$. Now we have $G \leq a+b+a = (a+b+c)-(c-a) = 5n - (c - a) \leq 5n - n = 4n$.

Case 2b. $b \neq c$ and $c - a < n$. Since $a \leq b \leq c$ we now also have $c - b < n$. Moreover, we have $b \neq c$ and therefore $a \neq c$. So $\gcd(c, a) = \gcd(c - a, a) \leq c - a < n$ and $\gcd(b, c) = \gcd(b, c - b) \leq c - b < n$. Also note that $a \leq \frac{5}{3}n$. Therefore $G \leq \frac{5}{3}n + n + n < 4n$.

So in all cases we have $G \leq 4n$. We can achieve $G = 4n$ via $a = n$ and $b = c = 2n$, since then we have $\gcd(a, b) = \gcd(c, a) = n$ and $\gcd(b, c) = 2n$.

Therefore the maximum value of G is $5n$ if $3 \mid n$ and $4n$ if $3 \nmid n$. □



4. Let G' be the intersection of the line BF and the circumcircle $\triangle BCD$. We show that $DEFG'$ is a cyclic quadrilateral and that $G = G'$.

Since $EF \parallel BC$ we have $180° - \angle DEF = \angle AEF = \angle ACB$, and since $BDCG'$ is a cyclic quadrilateral we have $\angle ABC = \angle DCB = \angle DG'B = \angle DG'F$, therefore $180° - \angle DEF = \angle DG'F$. Therefore $DEFG'$ is a cyclic quadrilateral. It follows that we have $\angle DEG' = \angle DFG' = \angle DFB$. Since $ADBF$ is a cyclic quadrilateral we have $\angle DFB = \angle DAB$, so $\angle DEG' = \angle DAB$, and therefore $EG' \parallel AB$. As G' lies on the circumcircle of $\triangle BCD$, we deduce that $G = G'$. □

IMO Team Selection Test 2, May 2019

Problems

1. In each of the different grades of a high school there are an odd number of pupils. Each pupil has a best friend (who possibly is in a different grade). Everyone is the best friend of their best friend. In the upcoming school trip, every pupil goes to either Rome or Paris. Show that the pupils can be distributed over the two destinations in such a way that

(i) every student goes to the same destination as their best friend;

(ii) for each grade the absolute difference between the number of pupils that are going to Rome and that of those who are going to Paris is equal to 1.

2. Determine all 4-tuples (a, b, c, d) of positive real numbers satisfying $a + b + c + d = 1$ and

$$\max\left(\frac{a^2}{b}, \frac{b^2}{a}\right) \cdot \max\left(\frac{c^2}{d}, \frac{d^2}{c}\right) = (\min(a + b, c + d))^4.$$

3. Let ABC be an acute triangle with circumcentre O. Point Q lies on the circumcircle of $\triangle BOC$ and OQ is a diameter of this circle. Point M lies on CQ and point N lies on the interior of the line segment BC in such a way that $ANCM$ is a parallelogram. Show that the circumcircle of $\triangle BOC$ and the lines AQ and NM are concurrent.

4. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

- $f(p) > 0$ for all prime numbers p,
- $p \mid \big(f(x) + f(p)\big)^{f(p)} - x$ for all $x \in \mathbb{Z}$ and all prime numbers p.

Solutions

1. We call a pupil a *singleton* if their best friend is in a different grade.

First give all singletons a destination as follows. Start by choosing any singleton A_1 and send them to Paris together with their best friend A_2. Suppose the destinations for pupils $A_{2m+1}, A_{2m+2}, \ldots, A_{2k}$, with $m \leq 0$ and $k \geq 1$ are chosen. If possible, first choose in the grade of A_{2k} a singleton A_{2k+1} who doesn't have a destination yet, and send them to the city different from the destination of A_{2k}, together with their best friend A_{2k+2}. If possible, then choose in the grade of A_{2m+1} a singleton A_{2m} who doesn't have a destination yet, and send them to the city different from the destination of A_{2m+1}, together with their best friend A_{2m-1}. Continue this process until pupils $A_{2m+1}, A_{2m+2}, \ldots, A_{2k}$ with $m \leq 0$ and $k \geq 1$ have destinations assigned to them and there are no singletons in the grades of A_{2m+1} and A_{2k} who don't have a destination yet. These pupils $A_{2m+1}, A_{2m+2}, \ldots, A_{2k}$ are by construction such that A_{2i-1} and A_{2i} are best friends and A_{2i} and A_{2i+1} have different destinations.

In every grade except that of A_{2m+1} and A_{2k} the same number of pupils is assigned to each destination. In the grades of A_{2m+1} and A_{2k} (which are different: otherwise no other singletons are in this grade, and all other pupils form pairs of best friends, making the number of pupils in that class even; contradiction!) one destination is assigned one more pupil than the other, and these grade no longer have any singletons without destination.

Repeat this process until every singleton has a destination. All remaining pupils now are in the same grade as their best friends, so every grade has an even number of pupils without destination and therefore an odd number of pupils with destination. So every grade contains a pupil on an end of exactly one of the sequences of pupils (which then must be the last time a pupil of this grade occurs in such a sequence). Therefore for each grade, one destination is assigned exactly one more pupil of that grade than the other.

Now consider any grade, and suppose that the number of singletons assigned to Paris is one more than that assigned to Rome. Give the pupils in this grade whose best friends are also in this grade a destination as follows. If the number of pairs of best friends is odd, first send one pair to Rome; otherwise do nothing. From the remaining even number of pairs of best friends, send half of them to Paris and half of them to Rome.

Now note that in the first step, if the number of pairs was odd, then Rome is assigned one more pupil than Paris. Therefore in any grade, the absolute difference of the number of students going to Paris and the number going to Rome is equal to 1. $\square$

2. We assume that $a+b \le c+d$. As we can simultaneously interchange a, b and c, d without changing the problem, this assumption will not give any loss of generality. We also assume that $a \le b$ and $c \le d$; no generality is lost here as we can interchange a and b (resp. c and d) without changing the problem. Now we have $a^3 \le b^3$, so $\frac{a^2}{b} \le \frac{b^2}{a}$, and analogously we have $\frac{c^2}{d} \le \frac{d^2}{c}$. The given equation is now equivalent to

$$\tfrac{b^2}{a} \cdot \tfrac{d^2}{c} = (a+b)^4.$$

As $\frac{b^2}{a} \ge b$ and $\frac{d^2}{c} \ge d$, the left hand side is at least bd. We now show that $bd \le (a+b)^4$, from which follows that equality must hold everywhere.

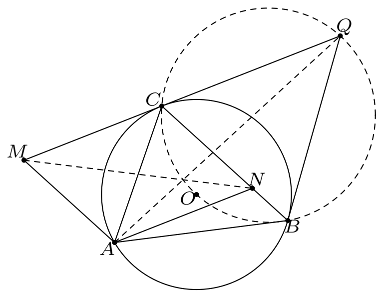
Since $a \le b$ we have $b \ge \frac{1}{2}(a+b)$, and analogously we have $d \ge \frac{1}{2}(c+d)$. Since $c+d \ge a+b$, it follows that $d \ge \frac{1}{2}(a+b)$. Moreover, from $a+b+c+d=1$ and $a+b \le c+d$ we deduce that $a+b \le \frac{1}{2}$, so

$$bd \le \tfrac{1}{4}(a+b)^2 \le (a+b)^2(a+b)^2 = (a+b)^4.$$

Therefore, each inequality used in the above must actually be an equality. Since we have used $\frac{b^2}{a} \ge b$ it follows that $a=b$, and analogously that $c=d$, and since we have used $a+b \le \frac{1}{2}$ it follows that $a+b=\frac{1}{2}$. Therefore $a=b=c=d=\frac{1}{4}$.

So the only possible solutions is $(a,b,c,d) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Substitution of this candidate solution gives $\frac{1}{16}$ on both sides of the equation, and therefore actually is a solution. □

3. Let T be the intersection of the circumcircle of $\triangle BOC$ with AQ. We show that T lies on NM. Write $\alpha = \angle BAC$. Then we have $\angle BOC = 2\alpha$ by the inscribed angle theorem. Since $|OB| = |OC|$ and $\angle OCQ = 90° = \angle OBQ$ by Thales's theorem, we have $\triangle OBQ \cong \triangle OCQ$. Hence we have $\angle COQ = \angle BOQ = \alpha$.



As $ANCM$ is a parallelogram, we see that $\angle AMC = 180° - \angle MCN = \angle QCN = \angle QCB = \angle QOB = \alpha$. Hence we have $\angle CNA = \angle AMC = \alpha$. Now we see that $\angle QTB = \angle QOB = \alpha = \angle CNA$, from which follows that $\angle ATB = 180° - \angle QTB = 180° - \angle CNA = \angle ANB$. Hence $ATNB$ is a cyclic quadrilateral. Also note that $ATCM$ is a cyclic quadrilateral since $\angle AMC = \alpha = \angle COQ = \angle CTQ = 180° - \angle ATC$.

As $ATCM$ is a cyclic quadrilateral, we have $\angle ATM = \angle ACM = 180^\circ - \angle QCB - \angle BCA$, using supplementary angles at C. Recall that $\angle QCB = \angle QOB = \alpha = \angle CAB$, so using the sum of the angles in triangle ABC, we obtain $\angle ATM = 180^\circ - \angle CAB - \angle BCA = \angle ABC$. Since $ATNB$ is a cyclic quadrilateral, we see that $\angle ABC = \angle ABN = 180^\circ - \angle ATN$, so we have $\angle ATM = 180^\circ - \angle ATN$. Therefore M, T, and N are collinear, as we have set out to prove. $\square$

4. Suppose that f is a function satisfying both relations.

Substituting $x = p$ gives $p \mid \big(2f(p)\big)^{f(p)} - p$ and as p is prime, we have $p \mid 2f(p)$. So either $p = 2$ or $p \mid f(p)$.

Substituting $x = 0$ gives $p \mid \big(f(0) + f(p)\big)^{f(p)} - 0$ and as p is prime, we have $p \mid f(0) + f(p)$. Since $p \mid f(p)$ if $p \neq 2$, it follows that $p \mid f(0)$ if $p \neq 2$. So $f(0)$ is divisible by infinitely many prime numbers and therefore must be equal to 0. From $2 \mid f(0) + f(2)$ we now see that $2 \mid f(2)$, so $p \mid f(p)$ for all prime numbers p.

Now the second of the given relations translates to $f(x)^{f(p)} \equiv p \mod p$ for all integers x and prime numbers p. It follows that $p \mid f(x)$ if and only if $p \mid x$. Applying this observation to the case in which x is a prime number $q \neq p$, then we see that $f(q)$ is not divisible by any prime number $p \neq q$. As $f(q) > 0$, we deduce that $f(q)$ is a power of q. Fermat's Little Theorem states that for all prime numbers p and integers n we have $n^p \equiv n \mod p$, so we also have $n^{p^t} \equiv n \mod p$ for all non-negative integers t. As $f(p)$ is of the form p^t with t a non-negative integer for all prime numbers p, we deduce from the second given relation that $f(x) \equiv x \mod p$ for all integers x and prime numbers p. Thus $p \mid f(x) - x$ for all integers x and all prime numbers p.

Therefore, for any fixed $x \in \mathbb{Z}$ the integer $f(x) - x$ is divisible by infinitely many prime numbers and therefore equal to 0. Hence $f(x)$ must be equal to x for all integers x.

Now suppose that $f(x) = x$ for all integers x. Then $f(p) > 0$ for all prime numbers p and

$$\big(f(x) + f(p)\big)^{f(p)} - x = (x+p)^p - x \equiv x^p - x \equiv 0 \mod p$$

for all integers x and all prime numbers p by Fermat's Little Theorem.

It follows that $f(x) = x$ is the unique function $\mathbb{Z} \to \mathbb{Z}$ satisfying both relations. $\square$

IMO Team Selection Test 3, May 2019

Problems

1. In a cyclic quadrilateral $ABCD$, the intersection of the diagonals is called E. A line through E, not equal to AC or BD, intersects AB in P and BC in Q. The circle that is tangent to PQ in E and also goes through D, intersects the circumcircle of $ABCD$ another time in the point R. Prove that B, P, R, and Q lie on a circle.

2. Let n be a positive integer. Prove that $n^2 + n + 1$ cannot be written as the product of two positive integers of which the difference is smaller than $2\sqrt{n}$.

3. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the following two conditions:

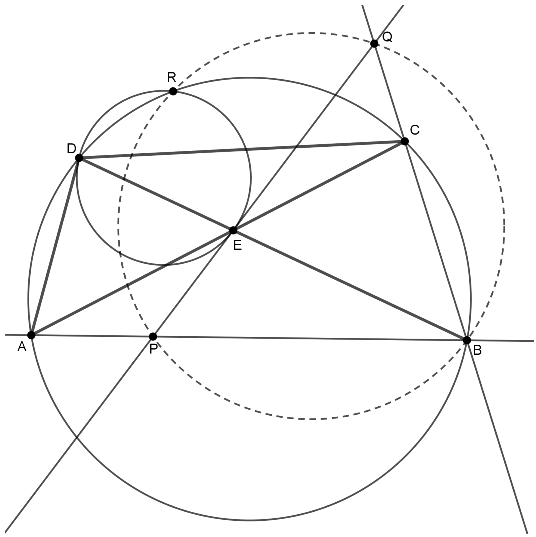
(i) for all integers x we have $f(f(x)) = x$;

(ii) for all integers x and y such that $x + y$ is odd, we have $f(x) + f(y) \geq x + y$.

4. There are 300 participants to a mathematics competition. After the competition some of the contestants play some games of chess. Each two contestants play at most one game against each other. There are no three contestants, such that each of them plays against each other. Determine the maximum value of n for which it is possible to satisfy the following conditions at the same time: each contestant plays at most n games of chess, and for each m with $1 \leq m \leq n$, there is a contestant playing exactly m games of chess.

Solutions

1. We consider the configuration in which P lies on the interior of AB and Q does not lie on the interior of BC; other configurations can be solved analogously.



Because $DBCR$ is cyclic, we have $\angle QCR = 180° - \angle BCR = \angle BDR$. As EQ is tangent to the circle through E, D, and R we have $\angle BDR = \angle EDR = \angle QER$, hence $\angle QCR = \angle QER$. This implies that $ERQC$ is a cyclic quadrilateral. Analogously, $EPAR$ is also cyclic.

We now get $\angle PRQ = \angle PRE + \angle ERQ = \angle PAE + 180° - \angle ECQ = \angle BAE + \angle ECB$. Because of the sum of the angles in $\triangle ABC$ this equals $180° - \angle ABC = 180° - \angle PBQ$. Hence $\angle PRQ = 180° - \angle PBQ$, and hence $BPRQ$ is a cyclic quadrilateral. $\square$

2. Suppose that a and b are positive integers satisfying $ab = n^2 + n + 1$. We will prove that $|a - b| \geq 2\sqrt{n}$. Note that $(a - b)^2 \geq 0$. Adding $4ab$ to both sides yields

$$(a+b)^2 = (a-b)^2 + 4ab \geq 4ab = 4n^2 + 4n + 4 > 4n^2 + 4n + 1 = (2n+1)^2.$$

As $(a + b)^2$ is a square, we have $(a + b)^2 \geq (2n + 2)^2$. Then it follows that

$$(a - b)^2 = (a + b)^2 - 4ab \geq (2n + 2)^2 - 4(n^2 + n + 1) = 4n$$

hence $|a - b| \geq 2\sqrt{n}$. $\square$

3. The function $f(x) = x$ for all x, satisfies the conditions. Now suppose that $f(x) = x$ does not hold for all x. As of (i), there is both a value of x for which $f(x) > x$ and a value of x for which $f(x) < x$. Let $a \in \mathbb{Z}$ be such that $f(a) < a$ and consider an arbitrary x with $x \not\equiv a \mod 2$. Then we have $f(x) + f(a) \geq x + a$, hence $f(x) \geq x + a - f(a) > x$. Write $w = f(x)$, then we have $f(w) = f(f(x)) = x < f(x) = w$. If $w \not\equiv a \mod 2$, then we would have $f(w) > w$, which gives a contradiction. Hence $f(x) = w \equiv a \mod 2$ and also $f(x) \not\equiv x \mod 2$.

Let x_1 and x_2 both be of different parity from a. Then they both do not have the same parity as their function values, hence $x_1 + f(x_2)$ and $x_2 + f(x_1)$ are odd. By substituting $x = x_1$, $y = f(x_2)$, and also $x = x_2$, $y = f(x_1)$, we find

$$f(x_1)+x_2 = f(x_1)+f(f(x_2)) \geq x_1+f(x_2) = f(f(x_1))+f(x_2) \geq f(x_1)+x_2.$$

Hence, equality must hold everywhere and $f(x_1) + x_2 = x_1 + f(x_2)$. Fix x_1 and write $c = f(x_1) - x_1$, then we have $f(x_2) = x_2 + c$ for all x_2 not having the same parity as a. Moreover, we know that c must be odd. If $z = x_2 + c$, then z can take all possible values not having the same parity as x_2 (and thus having the same parity as a). We have

$$f(z) = f(x_2 + c) = f(f(x_2)) = x_2 = z - c.$$

Now write $d = c$ if a is odd and $d = -c$ if a is even, then the function satisfies

$$f(x) = \begin{cases} x + d & \text{if } x \text{ is even,} \\ x - d & \text{if } x \text{ is odd.} \end{cases}$$

We check functions of this shape for any arbitrary odd d. We see that $f(x)$ and x never have the same parity, hence $f(f(x)) = x + d - d = x$, which means (i) is satisfied. Moreover, if $x + y$ is odd, then x and y have different parities, hence $f(x) + f(y) = x + y + d - d = x + y \geq x + y$, which means (ii) is satisfied as well.

We conclude that the functions of this form together with the function $f(x) = x$ for all x, are the solutions. □

4. We will prove that the maximum value of n is 200. We will first give an example with $n = 200$. Consider players $A_1, A_2, \ldots, A_{200}$ and $B_1, B_2, \ldots, B_{100}$. These are 300 players in total. Let B_i play a game against the players A_j with $1 \leq j \leq i + 100$, and assume no other contestants play a game against each other. Then B_i has exactly $100 + i$ opponents, so for $101 \leq m \leq 200$ there is a contestant playing exactly m games of chess. Moreover, for $j > 100$, the player A_j is playing against the contestants B_i with $i \geq j - 100$; these are $100 - (j - 101) = 201 - j$ players. Because j can vary from 101 up to and including 200, the number of opponents varies from 100 up to and including 1. Hence, for $1 \leq m \leq 100$ there is also a contestant playing exactly m games of chess. Lastly, for $j \leq 100$, contestant A_j plays against all B_i; these are 100 opponents. Hence, there is no contestant playing more than 200 games of chess. We see that this example meets the requirements for $n = 200$.

Now we will prove that $n > 200$ cannot be realised. We prove this by contradiction, so assume that $n > 200$. Then in any case there is a contestant A playing exactly 201 games of chess, against players $B_1, B_2, \ldots, B_{201}$. In total, there are 300 contestants, so besides them and A, there are another $300 - 1 - 201 = 98$ contestants, who we will call $C_1, \ldots, C_{98}$. If a contestant B_i is playing a game against another contestant B_j, then together with A they form a triple that plays three games among each other; this is not allowed. Hence B_i does not play against any of the other B_j. Therefore, they play at most $1 + 98 = 99$ games. This means that the contestants who play against exactly m other contestants with $100 \leq m \leq 200$, must all be among the contestants C_i. However, these are only 98 distinct contestants, which gives a contradiction. We conclude that $n > 200$ is impossible and that $n = 200$ is the maximum.

Junior Mathematical Olympiad, September 2018

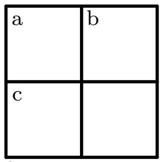
Problems

Part 1

1. When writing down the date 12 August 2018 using eight digits, each digit occurs exactly twice: 12-08-2018. There are more dates in 2018 having the same property. How many dates in 2018 have this property, including 12 August?

A) 5 B) 6 C) 7 D) 8 E) 9

2. In the crossword puzzle on the right, each square has to be filled with one of the digits 1 to 9. A digit may occur multiple times. For the 2-digit numbers formed in the rows and columns we are given the following four hints:



Across	**Down**
a. An odd number	a. A square
c. A square	b. An odd number

The puzzle has more than one solution, but the digit in the top-left corner is always the same. Which digit is in the top-left corner?

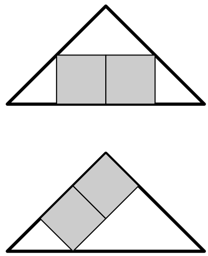
A) 1 B) 2 C) 3 D) 4 E) 6

3. Sophie likes to wear red and blue T-shirts. She decided to wear either a red or a blue T-shirt each day, starting from 1 January 2019. She does not want to say which colour she will be wearing on 1 and 2 January. From 3 January on, she will choose the colour of her T-shirt each day according to the following rule: she chooses red if she wore two different colours the last two days, and she chooses blue if she wore the same colour the last two days.

By following this rule, she will wear a blue T-shirt on her birthday, 14 January. Is it possible to determine with certainty the colours of the T-shirts she will be wearing on 28 and 29 January?

A) 28th: red, 29th: blue
B) 28th: blue 29th: blue
C) 28th: blue 29th: red
D) 28th: could be either 29th: blue
E) 28th: blue 29th: could be either

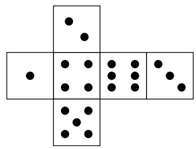
4. We have an isosceles triangle with two angles of 45 degrees; the long side has length 1. Within the triangle we put two squares of the same size. We can do this in two ways depicted in the two figures.



What is the total area of the two squares in the top figure minus the total area of the two squares in the bottom figure?

A) $\frac{1}{72}$ B) $\frac{1}{48}$ C) $\frac{1}{36}$ D) $\frac{1}{24}$ E) $\frac{1}{18}$

5. On a glass table, we arrange 100 dice tightly in a 10 by 10 square. This is done in such a way that if two dice are touching each other along a face, these faces have the same number of pips. Both the top and bottom faces of the 100 dice are visible. Altogether, on the front, rear, left, and right side there are 40 dice faces visible. We add the pips on all of these 240 visible faces.

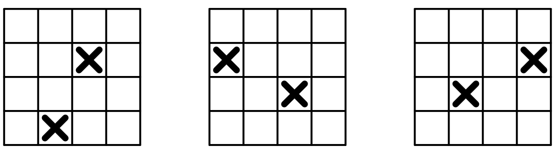


What is the largest outcome that we can get?

In the figure on the right, you can see a net representing a dice.

A) 840 B) 880 C) 920 D) 1240 E) 1440

6. A large square is subdivided into 16 squares. We put crosses in two of these squares (see the figure). This can be done in different ways. Sometimes two such ways only differ by a rotation, for example the left two squares below. In this case, we consider the two ways as the same one and only count it once.

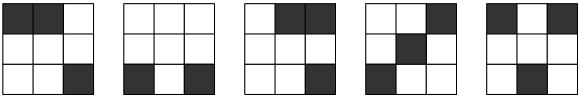


Pay attention: the right two squares are not considered to be the same. You could obtain the right square from the middle one by reflection, but not by using a rotation!

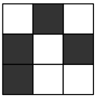
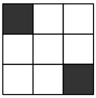
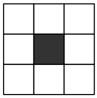
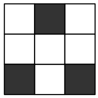
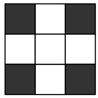
In how many different ways can you put the two crosses?

A) 21 B) 30 C) 32 D) 34 E) 36

7. Using 27 small cubes, each coloured black or white, we build a $3{\times}3{\times}3$ cube. This large cube has six views: a front, rear, left, right, top, and bottom view. In the figure, five of the views of the large cube are depicted.



Which could be the sixth view of the large cube?

A)  B)  C)  D)  E) 

8. How many distinct pairs of digits a and b are there such that $5a68 \times 865b$ is divisible by 824?
Pay attention: we are counting pairs, so for example if for $a = 0$ the values $b = 0$, $b = 1$, and $b = 2$ are all valid, then these are counted as three different pairs.

A) 10 B) 11 C) 15 D) 19 E) 21

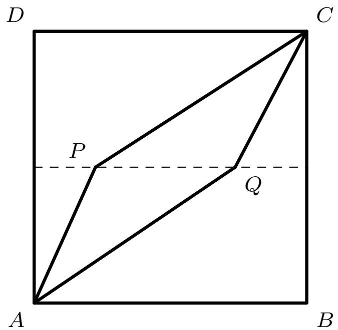
Part 2

1. Anne, Bert, Christiaan, Dirk, and Eveline are particpating in a chess tournament. They find out that their average age is exactly 28 years. Exactly one year later Anne, Bert, Christiaan, and Dirk participate together with Freek in the tournament. This time, their average age is exactly 30 years.
How many years older is Freek compared to Eveline?

2. What is the smallest positive integer x for which the outcome of $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} + \frac{x}{6}$ is an integer?

3. For each of the two fractions $\frac{2018}{2011}$ and $\frac{2054}{2019}$ we subtract the same integer a from both the numerator and the denominator. The two fractions we get, are equal.
What integer is a?

4. The sides of the square $ABCD$ have length 10. The points P and Q lie on the line connecting the midpoints of AD and BC. If we connect P with A and C, and also Q with A and C, then the square is divided into three parts having equal area.
What is the length of PQ? (*The figure is not drawn to scale.*)

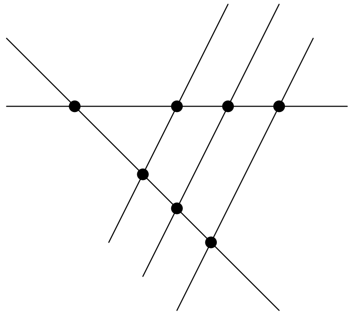


5. Jan has written the number A in two different ways as a fraction. Unfortunately, two numbers have become invisible due to spilled ink spots:

$$A = \frac{\blacksquare + 3}{12} = \frac{15}{26 - \blacksquare}.$$

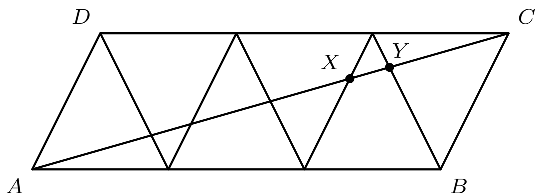
Lia has found Jan's note and wants to find all possibilities for the number A. She knows that the numbers underneath the ink spots must be positive integers, but the number A does not have to be an integer. Because she does not know which numbers are hidden below the ink spots, she searches for all combinations for which the equality holds. In this way, she finds multiple possible values for A. The largest one she calls A_{max} and the smallest one she calls A_{min}.
Determine $A_{\text{max}} - A_{\text{min}}$.

6. Tim draws five straight lines in a plane. The lines are continuing indefinitely, and there are no three lines going through the same point. For each intersection point of two lines he gets one sweet, and for each set of two or more parallel lines he gets one sweet as well. For instance, in the example below, he gets 8 sweets because firstly there are 7 intersection points, and secondly there is 1 set of three parallel lines, which is worth another sweet.



What are all possible numbers of sweets that Tim can get?

7. Six equilateral and equally sized triangles are glued to form a parallelogram $ABCD$, see the figure.



The length of AC is 10. What is the length of XY?
Attention: the figure is not drawn to scale.

8. In a class room there are a number of students. They find out that for each triplet of students, the following two statements are both true:

- Two of them never wrote a report together.
- Two of them did once write a report together.

What is the maximum possible number of students in the class room?

Answers

Part 1

1. B) 6

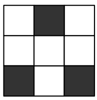
2. E) 6

3. D) 28th: could be either, 29th: blue

4. A) $\frac{1}{72}$

5. C) 920

6. C) 32

7. D) 

8. D) 19

Part 2

1. 5 years

2. 20

3. 2009

4. $\frac{20}{3}$ $(= 6\frac{2}{3})$

5. $\frac{57}{4}$ $(= 14\frac{1}{4})$

6. $1, 5, 8, 10$

7. $\frac{5}{6}$ $(= \frac{10}{12})$

8. 5

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