1st BENELUX MATHEMATICAL OLYMPIAD
Bergen op Zoom (Netherlands)
May 9, 2009

Language: **English**

Problem 1. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that satisfy the following two conditions:

- $f(n)$ is a perfect square for all $n \in \mathbb{Z}_{>0}$;
- $f(m+n) = f(m) + f(n) + 2mn$ for all $m, n \in \mathbb{Z}_{>0}$.

Problem 2. Let n be a positive integer and let k be an odd positive integer. Moreover, let a, b and c be integers (not necessarily positive) satisfying the equations

$$a^n + kb = b^n + kc = c^n + ka.$$

Prove that $a = b = c$.

Problem 3. Let $n \geq 1$ be an integer. In town X there are n girls and n boys, and each girl knows each boy. In town Y there are n girls, g_1, g_2, …, g_n, and $2n-1$ boys, b_1, b_2, …, b_{2n-1}. For $i = 1, 2, \ldots, n$, girl g_i knows boys b_1, b_2, …, b_{2i-1} and no other boys. Let r be an integer with $1 \leq r \leq n$. In each of the towns a party will be held where r girls from that town and r boys from the same town are supposed to dance with each other in r dancing pairs. However, every girl only wants to dance with a boy she knows. Denote by $X(r)$ the number of ways in which we can choose r dancing pairs from town X, and by $Y(r)$ the number of ways in which we can choose r dancing pairs from town Y.
Prove that $X(r) = Y(r)$ for $r = 1, 2, \ldots, n$.

Problem 4. Given trapezoid $ABCD$ with parallel sides AB and CD, let E be a point on line BC outside segment BC, such that segment AE intersects segment CD. Assume that there exists a point F inside segment AD such that $\angle EAD = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, and assume that K is different from I and J.
Prove that K belongs to the circumcircle of $\triangle ABI$ if and only if K belongs to the circumcircle of $\triangle CDJ$.

Time allowed: 4 hours and 30 minutes
Each problem is worth 7 points