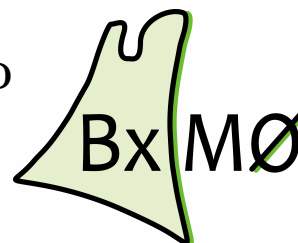


1st BENELUX MATHEMATICAL OLYMPIAD  
Bergen op Zoom (Netherlands)  
May 9, 2009



Language: **English**

**Problem 1.** Find all functions  $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$  that satisfy the following two conditions:

- $f(n)$  is a perfect square for all  $n \in \mathbb{Z}_{>0}$ ;
- $f(m+n) = f(m) + f(n) + 2mn$  for all  $m, n \in \mathbb{Z}_{>0}$ .

**Problem 2.** Let  $n$  be a positive integer and let  $k$  be an odd positive integer. Moreover, let  $a, b$  and  $c$  be integers (not necessarily positive) satisfying the equations

$$a^n + kb = b^n + kc = c^n + ka.$$

Prove that  $a = b = c$ .

**Problem 3.** Let  $n \geq 1$  be an integer. In town  $X$  there are  $n$  girls and  $n$  boys, and each girl knows each boy. In town  $Y$  there are  $n$  girls,  $g_1, g_2, \dots, g_n$ , and  $2n - 1$  boys,  $b_1, b_2, \dots, b_{2n-1}$ . For  $i = 1, 2, \dots, n$ , girl  $g_i$  knows boys  $b_1, b_2, \dots, b_{2i-1}$  and no other boys. Let  $r$  be an integer with  $1 \leq r \leq n$ . In each of the towns a party will be held where  $r$  girls from that town and  $r$  boys from the same town are supposed to dance with each other in  $r$  dancing pairs. However, every girl only wants to dance with a boy she knows. Denote by  $X(r)$  the number of ways in which we can choose  $r$  dancing pairs from town  $X$ , and by  $Y(r)$  the number of ways in which we can choose  $r$  dancing pairs from town  $Y$ . Prove that  $X(r) = Y(r)$  for  $r = 1, 2, \dots, n$ .

**Problem 4.** Given trapezoid  $ABCD$  with parallel sides  $AB$  and  $CD$ , let  $E$  be a point on line  $BC$  outside segment  $BC$ , such that segment  $AE$  intersects segment  $CD$ . Assume that there exists a point  $F$  inside segment  $AD$  such that  $\angle EAD = \angle CBF$ . Denote by  $I$  the point of intersection of  $CD$  and  $EF$ , and by  $J$  the point of intersection of  $AB$  and  $EF$ . Let  $K$  be the midpoint of segment  $EF$ , and assume that  $K$  is different from  $I$  and  $J$ .

Prove that  $K$  belongs to the circumcircle of  $\triangle ABI$  if and only if  $K$  belongs to the circumcircle of  $\triangle CDJ$ .

*Time allowed: 4 hours and 30 minutes  
Each problem is worth 7 points*