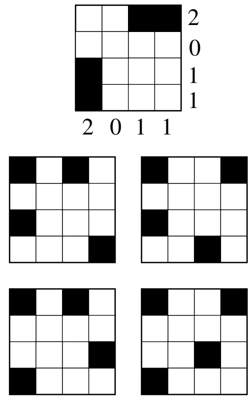
First round
Dutch Mathematical Olympiad

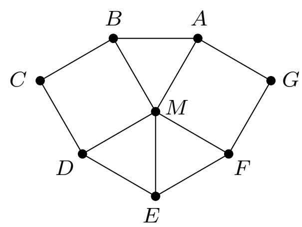
Friday 4 Februari 2011

Solutions



A1. D) 5 Notice that all squares in the second row and second column must be white. We consider two cases, depending on the color of the upper left square.
If this square is white, then the last two squares in the first row and column must be black. This determines the coloring. See the top figure.
If this square is black, both the frist row and first column require exactly one more black square. For each of the resulting $2 \times 2 = 4$ choices, there is exactly one solution. Indeed, one row and one column are left that need an additional black square. Therefore the square at the intersection of this row and column must be colored black, and the remaining squares must be colored white, see bottom four figures.

A2. C) June The year of the date we are looking for, starts with a digit 2 or higher. We will look for the first date of which the year starts with digit 2, and all eight digits are different. If such a date exists, we are done.
For the month, both 11 (two equal digits) and 12 (digit 2 is already used) can be rejected. Therefore, the month (01 to 10) contains digit 0. This implies that the day starts with digit 1 or 3. In the second case, it's the 31st, since digit 0 is already taken. In both cases, the day contains digit 1. Both digit 0 and 1 being taken, the smallest possible year is 2345. The smallest number we can use for the month is then 06, that is, June. Finally, the day will be the 17th. Observe that the constructed date 17-06-2345 consists of eight different digits, as required.



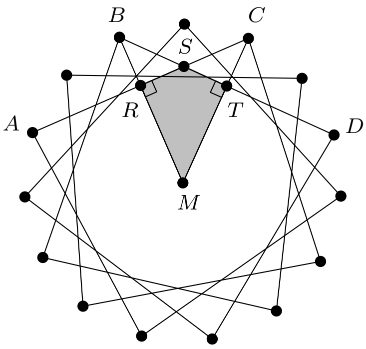
A3. B) $8 + 3\sqrt{3}$ The heptagon can be partitioned into two squares and three equilateral triangles, all with sides of length 2. We know that each of the squares has an area of 4. Using the Pythagorean theorem, we can compute the height of triangle ABM to be $\sqrt{2^2 - 1} = \sqrt{3}$. Hence, the area of the triangle equals $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$. Summing up the areas of the squares and triangles, we arrive at $2 \cdot 4 + 3 \cdot \sqrt{3} = 8 + 3\sqrt{3}$ for the area of the heptagon.

A4. A) only Alice As Brian's prediction was wrong, he has at least six correct answers. Alice's prediction was wrong as well, which implies that Brian answered at the most one more question correctly than Alice. Hence, Alice has at least five correct answers. Since Carl made a wrong prediction, he answered more questions correctly than Alice, hence at least six. Alice cannot have more thatn five correct answers. Indeed, then Carl would have seven correct answers, leading to a total of at least $6 + 6 + 7 = 19$ correct answers, as the teacher (incorrectly) predicted. We can conclude that Alice answered five questions correctly. Since the others have at least six correct answers, Alice is the only one with the smallest number of correct answers.

A5. C) 62 Jack can certainly write down 62 numbers, for example: the numbers from 1 to 62 (since $62 + 61 < 125$). More than 62 numbers will not be possible. Indeed, the numbers from 25 to 100 can be partitioned into pairs of sum 125: $25 + 100 = 125$, $26 + 99 = 125$, and so on up to $62 + 63 = 125$. Jack must skip at least one number from each of the 38 pairs. In total, therefore, he can write down no more than $100 - 38 = 62$ of the numbers.

A6. B) 1 Using long division to divide $a = 11\cdots 11$ (2011 digits) by 37, you will quickly notice that 111 is divisible by 37. This is the fact that we will be using. It implies that the number $1110\cdots 0$ is divisible by 37, regardless of the number of trailing zeros. In particular, the following numbers are divisible by 37: $1110\cdots 0$ (2008 zeros), $1110\cdots 0$ (2005 zeros), $1110\cdots 0$ (2002 zeros), and so on up to 1110 (1 zero). The sum of these numbers equals $1\ldots 10$ (2010 digits 1), which is again divisible by 37. In conlusion, the remainder of a when divided by 37 is 1, since $a - 1$ is divisible by 37.

A7. C) 35 After 140 seconds, Ann has made 7 rounds and Bob only 5. At that moment, Ann leads by two full rounds. Hence after only $\frac{140}{4} = 35$ seconds, Anne leads by half a round. That is exactly the first moment she and Bob are at maximal distance from each other.

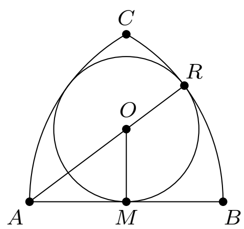


A8. B) 132° Denote the center of the 15-gon by M and the intersection of AC and BD by S (see figure). In quadrilateral $MRST$, we see that $\angle MRS = 90°$ and $\angle STM = 90°$. Furthermore, we see that $\angle TMR = \frac{2}{15} \cdot 360° = 48°$. As the angles of a quadrilateral sum to 360°, we find: $\angle RST = 360° - 2 \cdot 90° - 48° = 132°$. Observe that $\angle RST$ and $\angle BSC$ are opposite angles. Hence the sought-after angle also equals 132°.

B1. -1 We are given that $x = \frac{1}{1+x}$. Clearly, $x \neq 0$, since $0 \neq \frac{1}{1}$. Hence, on both sides of the equation, we may flip the numerator and denominator of the fraction. This results in: $\frac{1}{x} = 1 + x$. Combining both formulas, we obtain $x - \frac{1}{x} = x - (1 + x) = -1$.

B2. 20 Consider three escalators in a row: the first one going up, the second standing still, and the third going down. If Dion walkes up the first escalator, he arrives at the top after exactly 12 steps. Raymond, walking up the third escalator, takes 60 steps to reach the top and reaches only $\frac{1}{5}$ of the escalator after 12 steps. A third person, say Julian, takes the second escalator and walks at the same pace as Dion and Raymond. After 12 steps, he will be positioned exactly in between Dion and Julian, at $(\frac{5}{5} + \frac{1}{5})/2 = \frac{3}{5}$ of the escalator. Therefore, he will need $\frac{5}{3} \cdot 12 = 20$ steps to reach the top.

B3. 120 Let's call the eldest scout A. There are 5 possibilities for finding him a partner B for the first day. Then, there are 4 possible partners C for B on the second day, because he cannot be paired with A twice. Now for C, there are 3 possible partners D on the first day, since he cannot go with B again, and A is already paired. For D, there are now 2 possible partners E on the second day, since B and C are already paired, and A cannot be his partner because that would leave two scouts that are forced to form a pair on both days. Finally, there is one scout left. He has no choice but to team op with E on the first day, and with A on the second. In total there are $5 \times 4 \times 3 \times 2 = 120$ possibilities.



B4. $\frac{3}{8}$ Consider the inscribed circle. We denote its center by O and its radius by r. The points where the circle is tangent to AB and BC are denoted by M and R, respectively.
Since A, O and R are on a line, we have: $|AO| = |AR| - |OR| = 1 - r$. We also know that $|OM| = r$ and $|AM| = \frac{1}{2}|AB| = \frac{1}{2}$. Using the Pythagorean theorem, we find $|AM|^2 + |OM|^2 = |AO|^2$, and hence $\frac{1}{4} + r^2 = (1 - r)^2 = r^2 - 2r + 1$. This implies that $2r = \frac{3}{4}$ and therefore $r = \frac{3}{8}$.