We eat problems for breakfast.

Preferably unsolved ones...

52nd Dutch Mathematical Olympiad 2013

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2014 started with the first round in January 2013, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In total 32% more students than in 2012 participated in this first round: to be precise: 7424 students of 283 secondary schools.

Those 800 students from grade 5 (4, ≤ 3) that scored 24 (21, 18) points or more on the first round (out of a maximum of 36 points) were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

Those students from grade 5 (4, ≤ 3) that scored 26 (24, 21) points or more on the second round (out of a maximum of 40 points) were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total 149 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

Out of those 149, in total 143 participated in the final round on 13 September 2013 at Eindhoven University of Technology. This final round contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 52nd edition 2013.

The 34 most outstanding candidates of the Dutch Mathematical Olympiad 2013 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

Among the participants of the training programme, there were some extra girls, as this year we participated for the third time in the European Girls' Mathematical Olympiad (EGMO). In total there were eight girls competing to be in the EGMO team. The team of four girls was selected by a selection test, held on 21 March 2014. They attended the EGMO in Antalya, Turkey from 10 until 16 April, and the team returned with a gold and a bronze medal. For more information about the EGMO (including the 2014 paper), see www.egmo.org.

The same selection test was used to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Brugge, Belgium, from 2 until 4 May. The Dutch team managed to come first in the country ranking, and received two bronze medals, two silver medals and two gold medals. For more information about the BxMO (including the 2014 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2014 was selected by two team selection tests on 6 and 7 June 2014. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Cape Town, from 28 June until 6 July.

For younger students the Junior Mathematical Olympiad was held in October 2013 at the VU University Amsterdam. The students invited to participate in this event were the 70 best students of grade 1, grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2014 in South Africa consists of

- Tysger Boelens (18 years old)
  - bronze medal at BxMO 2013, gold medal at BxMO 2014
- Peter Gerlagh (17 years old)
  - bronze medal at BxMO 2011, honourable mention at BxMO 2012, gold medal at BxMO 2013
  - observer C at IMO 2012, bronze medal at IMO 2013
- Matthew Maat (14 years old)
  - bronze medal at BxMO 2014
- Michelle Sweering (17 years old)
  - bronze medal at EGMO 2012, silver medal at EGMO 2013, gold medal at EGMO 2014
  - honourable mention at IMO 2012, bronze medal at IMO 2013
- Bas Verseveldt (17 years old)
  - silver medal at BxMO 2012, bronze medal at BxMO 2013, gold medal at BxMO 2014
  - observer C at IMO 2013
- Jeroen Winkel (17 years old)
  - bronze medal at BxMO 2011, silver medal at BxMO 2012
  - observer C at IMO 2011, bronze medal at IMO 2012, silver medal at IMO 2013

We bring as observer C the promising young student

- Bob Zwetsloot (16 years old)
  - bronze medal at BxMO 2014

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Julian Lyczak (observer B), Utrecht University

First Round, January 2013

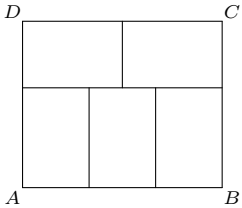
Problems

A-problems

A1. A traffic light is alternately green and red. The periods green and red are equally long and always of the same length: either 1, 2, or 3 minutes. There are four combinations for the colour of the light at 12:08 pm and at 12:09 pm: red–red, red–green, green–red, and green–green.
How many of these four combinations are possible, given that the light is red at 12:05 pm and also red at 12:12 pm?

A) 1 B) 2 C) 3
D) 4 E) The light cannot be red at both times.

A2. The rectangle $ABCD$ is divided into five equal rectangles. The perimeter of each of these small rectangles is 20.
What is the area of rectangle $ABCD$?



A) 72 B) 112 C) 120 D) 140 E) 150

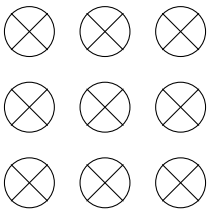
A3. The numbers a, b, c, d and e satisfy:

$$a+b+1 = b+c-2 = c+d+3 = d+e-4 = e+a+5.$$

Which is the largest of these five numbers?

A) a B) b C) c D) d E) e

A4. Nine light bulbs are put in a square formation. Each bulb can be either *on* or *off*. We can make a move by pressing a bulb. Then, the pressed bulb and the bulbs in the same row or column change their state from *on* to *off* or vice versa. Initially, all light bulbs are *on*.
What is the minimum number of moves needed to turn *off* all the light bulbs?



A) 3 B) 4 C) 5 D) 9 E) This is impossible.

A5. Out of a shipment of boxes, one fourth is empty. We open one fourth of all boxes and notice that one fifth of them is non-empty.
Which part of the unopened boxes is empty?

A) $\frac{4}{15}$ B) $\frac{1}{4}$ C) $\frac{1}{15}$ D) $\frac{1}{16}$ E) $\frac{1}{20}$

A6. A regular hexagon and an equilateral triangle have the same perimeter.
What is the ratio *area hexagon* : *area triangle*?

A) $2:3$ B) $1:1$ C) $4:3$ D) $3:2$ E) $2:1$

A7. What are the last four digits of 5^{2013}?

A) 0625 B) 2525 C) 3125 D) 5625 E) 8125

A8. Twenty students did a test. No two students answered the same number of questions correctly. Each question was answered correctly by at most three students.
What is the smallest number of questions that the test could have had?

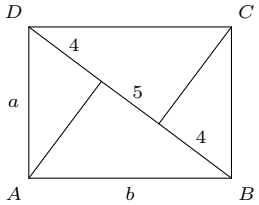
A) 63 B) 64 C) 67 D) 70 E) 71

B-problems

The answer to each B-problem is a number.

B1. What is the smallest positive integer consisting of the digits 2, 4 and 8, such that each digit occurs at least twice and the number is not divisible by 4?

B2. A rectangle $ABCD$ has sides of length a and b, where $a < b$. The lines through A and C perpendicular to the diagonal BD divide the diagonal into three segments of lengths 4, 5, and 4.
Calculate $\frac{b}{a}$.



B3. A bus calls at three stops. The middle bus stop is equally far from the first stop as from the last stop. Fred, standing at the middle bus stop, has to wait for 15 minutes for the bus to arrive. If he cycles to the first stop, he will arrive there at the same time as the bus. If instead he runs to the last stop, he will also arrive there at the same moment as the bus.
How long would it take Fred to cycle to the last stop and then run back to the middle stop?

B4. We write down the numbers from 1 to 30 000 one after the other to form a long string of digits:

$$123456789101112\ldots30000.$$

How many times does 2013 occur in this sequence?

Solutions

A-problems

A1. B) 2 It is given that the light is red at 12:05 pm. The colours at times 12:05 pm to 12:12 pm are now fixed for a traffic light of period 1: they are alternately red and green. When the period is 2, there are two possibilities and for period 3 there are three possibilities:

period	12:05	12:06	12:07	12:08	12:09	12:10	12:11	12:12
1 min	red	green	red	green	red	green	red	green
2 min	red	red	green	green	red	red	green	green
2 min	red	green	green	**red**	**red**	green	green	red
3 min	red	red	red	**green**	**green**	green	red	red
3 min	red	red	green	**green**	**green**	red	red	red
3 min	red	green	green	green	red	red	red	green

In three of the six cases, the light is red at 12:12 pm, as required. This gives us two colour combinations for the light at 12:08 pm and at 12:09 pm: red–red and green-green.

A2. C) 120 Twice the length of a small rectangle equals three times its width. Therefore, the ratio between length and width equals $3:2$. As the perimeter is 20, the length must be 6 and the width must be 4. We see that the area of each small rectangle equals $6 \times 4 = 24$, hence the area of rectangle $ABCD$ is $5 \times 24 = 120$.

A3. B) b Comparing sums of pairs of numbers, we find:

$$e + a < c + d < a + b < b + c < d + e.$$

Every sum of four of the numbers can be obtained by adding two of the pairs. For example, $a+b+c+e$ is equal to $(e+a)+(b+c)$. Of all four-tuples, a, c, d, e has the smallest sum, because $a + c + d + e = (e + a) + (c + d)$ is the sum of the two smallest pairs. The remaining number b must be the largest among the five numbers. Indeed, the largest number is the one for which the remaining four numbers have the smallest sum.

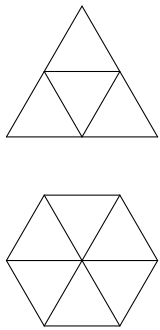
A4. A) 3 The order in which the moves are made is irrelevant for the final result. By pressing the three bulbs in the top row, all bulbs will change

from being *on* to being *off*. Indeed, the bulbs in the top row change their state three times, and the other bulbs change their state exactly once.

It is not possible to turn *off* all light bulbs by pressing two or fewer bulbs, because some bulb will not be in the same row or column as the chosen bulbs, and hence remain *on*.

A5. C) $\frac{1}{15}$ Without changing the problem, we may assume that there are 20 boxes. Hence, a total of 5 boxes is empty. Out of the 5 boxes that are opened, one fifth turns out to be non-empty, exactly 1 box. As 4 of the opened boxes are empty, there is exactly 1 empty box among the remaining 15 unopened boxes.

A6. D) $3:2$ We divide the hexagon into 6 equal equilateral triangles and divide the triangle into 4 equal equilateral triangles, see the figure. Since the hexagon and the triangle have the same perimeter, the sides of the triangle are twice as long as the hexagon's sides. Therefore, the triangles in both divisions have the same size. It follows that the ratio between the area of the hexagon and the area of the triangle equals $6:4$, or: $3:2$.



A7. C) 3125 We consider the last four digits of the powers of 5:

$$\begin{array}{rcl@{\qquad}rcr} 5^1 & = & 0005 & 5^5 & = & 3125 \\ 5^2 & = & 0025 & 5^6 & = & 1\,5625 \\ 5^3 & = & 0125 & 5^7 & = & 7\,8125 \\ 5^4 & = & 0625 & 5^8 & = & 39\,0625 \end{array}$$

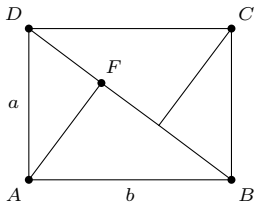
The last four digits of a power of 5 are already determined by the last four digits of the previous power of 5. For example: the last four digits of 5×390625 and 5×0625 are both 3125. Because the last four digits of 5^4 and 5^8 are the same, the last four digits of powers of 5 will repeat every four steps. The last four digits of 5^{2013} will be the same as those of 5^{2009} and those of 5^{2005}, continuing all the way down to the last four digits of $5^5 = 3125$. We conclude that 3125 are the last four digits of 5^{2013}.

A8. B) 64 Since every student answered correctly a different number of questions, at least $0 + 1 + 2 + \cdots + 19 = 190$ correct answers were given in

total. Because every question was answered correctly at most three times, there must be at least $\frac{190}{3} = 63\frac{1}{3}$ questions. That is, there were at least 64 questions.

B-problems

B2. $\frac{3}{2}$ Triangles ADF, BDA en BAF are similar (AA). Hence



$$\frac{b}{a} = \frac{|BA|}{|DA|} = \frac{|AF|}{|DF|} = \frac{|BF|}{|AF|}.$$

Therefore

$$\left(\frac{b}{a}\right)^2 = \frac{|AF|}{|DF|} \cdot \frac{|BF|}{|AF|} = \frac{|BF|}{|DF|} = \frac{9}{4}.$$

We conclude that $\frac{b}{a} = \frac{3}{2}$.

B3. 30 minutes The time needed by Fred to bike from the middle stop to the last stop and run back to the middle stop, is equal to the time he needs to bike from the middle stop to the first stop plus the time he needs to run from the middle stop to the last stop. This is because the distance to both stops is the same.

It is given that this amount of time equals the time the bus needs to get to the first stop plus the time it needs to get to the last stop. This is exactly twice the time the bus needs to get to the middle stop: $2 \times 15 = 30$ minutes.

B4. 25 times The combination of digits "2013" occurs 13 times as part of the following numbers: 2013, 12013, 22013, and 20130 to 20139. In addition, "2013" also occurs as the end of one number followed by the beginning of the next number. The different possibilities are:

2|013 does not occur, since no number starts with digit '0'.

20|13 occurs 11 times: 1320|1321 and 13020|13021 to 13920|13921.

201|3 occurs only once: 3201|3202, because no numbers larger than 30 000 were written down.

It is easy to verify that "2013" does not occur as a combination of three consecutive numbers. Therefore, "2013" occurs a total of $13 + 11 + 1 = 25$ times in the sequence of digits.

Second Round, March 2013

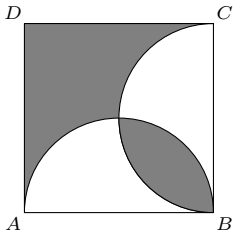
Problems

B-problems

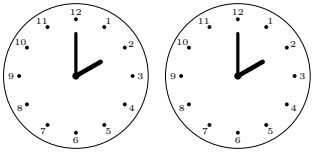
The answer to each B-problem is a number.

B1. A number of students took a test for which the maximum possible score was 100 points. Everyone had a score of at least 60 points. Exactly five students scored the maximum of 100 points. The average score among the students was 76 points.
What is the minimum number of students that could have taken the test?

B2. In the figure, a square $ABCD$ of side length 4 is given. Inside the square, two semicircles with diameters AB and BC are drawn.
Determine the combined area of the two grey shapes.



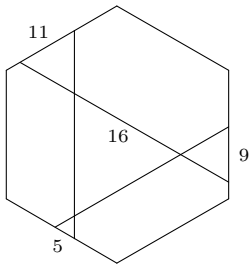
B3. Consider two clocks, like the ones in the figure on the right, whose hands move at a constant speed. Both clocks are defective; the hands of the first clock turn at a pace that is 1% faster than it should be, while the hands of the second clock turn at a pace that is 5% too fast. At a certain moment, both clocks show a time of exactly 2 o'clock. Some time passes until both clocks again show exactly the same time.
At that moment, what time do the clocks show?



B4. The number square on the right is filled with positive numbers. The *product* of the numbers in each row, in each column, and in each of the two diagonals is always the same.
What number is H?

$\frac{1}{2}$	32	A	B
C	2	8	2
4	1	D	E
F	G	H	16

B5. A regular hexagon is divided into seven parts by lines parallel to its sides, see the figure. Four of those pieces are equilateral triangles, whose side lengths are indicated in the figure.
What is the side length of the regular hexagon?



C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. We say a positive n-digit number ($n \geq 3$ and $n \leq 9$) is *above average* if it has the following two properties:

- the number contains each digit from 1 to n exactly once;
- for each digit, except the first two, the following holds: twice the digit is at least the sum of the two preceding digits.

For example, 31254 is above average because it consists of the digits 1 to 5 (each exactly once) and also

$$2 \cdot 2 \geq 3 + 1, \qquad 2 \cdot 5 \geq 1 + 2, \qquad \text{and} \qquad 2 \cdot 4 \geq 2 + 5.$$

(a) Give a 4-digit number that is above average and has '4' as its first digit.

(b) Show that no 4-digit number that is above average has '4' as its second digit.

(c) For 7-digit numbers that are above average, determine all possible positions of the digit '7'.

C2. We will call a triple (x, y, z) *good* if x, y, and z are positive integers such that $y \geq 2$ and the equation $x^2 - 3y^2 = z^2 - 3$ holds.
An example of a good triple is $(19, 6, 16)$, because $6 \geq 2$ and $19^2 - 3 \cdot 6^2 = 16^2 - 3$.

(a) Show that for every *odd* number $x \geq 5$ there are at least two good triples (x, y, z).

(b) Find a good triple (x, y, z) with x being *even*.

Solutions

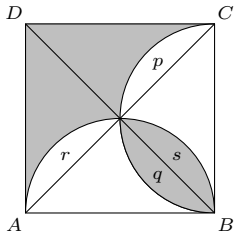
B-problems

B1. 13 A solution with thirteen students is possible. If five students scored 100 points and the remaining eights students scored 61 points, the average score equals $\frac{5\cdot 100+8\cdot 61}{13} = \frac{988}{13} = 76$ points, as required.
It is not possible that the number of students taking the test was twelve or less. Indeed, suppose that $n \leq 12$ students took the test. Five of them scored 100 points and the remaining $n-5$ students scored at least 60 points. Their total score would be at least $500 + (n-5)\cdot 60 = 60n + 200$. Their average score would then be at least

$$\frac{60n+200}{n} = 60 + \frac{200}{n} \geq 60 + \frac{200}{12} = 76\tfrac{2}{3},$$

since $n \leq 12$. This, however, contradicts the fact that their average score was 76.

B2. 8 Note that both circles go through the middle of the square. Therefore, the four circle segments indicated by p, q, r, and s all belong to one fourth of a circle with radius 2, hence the four segments have equal areas. Therefore, the combined area of the grey shapes equals the area of triangle ACD which is $\frac{1}{2}\cdot 4\cdot 4 = 8$.



B3. 5 o'clock In 12 hours, the amounts by which the hour hands of the first and the second clock are ahead increase by $\frac{1}{100}$ and $\frac{5}{100}$ of a turn respectively. Therefore, in 12 hours, the second clock increases its lead compared to the first clock by $\frac{5-1}{100} = \frac{1}{25}$ of a full turn. Hence after $25\cdot 12$ hours, the hour hand of the second clock has made exactly one extra full turn compared to that of the first clock. This is the first time the two clocks again display the same time.

During those $25\cdot 12$ hours, the hour hand of the first clock has made exactly $\frac{101}{100}\cdot 25 = 25\frac{1}{4}$ full turns. Both clocks will then display a time of $2+3=5$ o'clock.

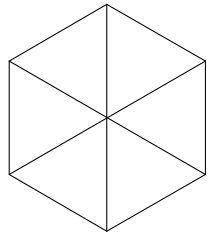
B4. $\frac{1}{4}$ The product of the eight numbers in the second and fourth row equals the product of the eights numbers in the first and second column. Writing this out, we get:

$\frac{1}{2}$	32	8	1
4	2	8	2
4	1	8	4
16	2	$\frac{1}{4}$	16

$$C \cdot 2 \cdot 8 \cdot 2 \cdot F \cdot G \cdot H \cdot 16 = \frac{1}{2} \cdot C \cdot 4 \cdot F \cdot 32 \cdot 2 \cdot 1 \cdot G.$$

Since C, F, and G are nonzero, we may divide both sides of the equation by C, F, and G. The resulting equation is $512 \cdot H = 128$, which implies that $H = \frac{1}{4}$. The figure shows one solution.

B5. 19 By dividing the regular hexagon into six equilateral triangles, we deduce that the length of the long diagonal AC of the hexagon equals twice the side length of the hexagon. We will compute three times the side length, namely $|AB| + |BC| + |CD|$. Observe that AB is a side of a parallelogram with the parallel side having length $11 + 16 = 27$. Thus we have $|AB| = 27$.



As triangle BCE is equilateral, we have $|BC| = |EB|$.

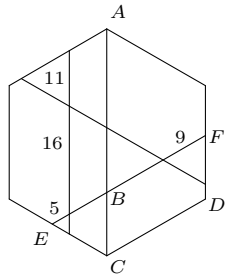
As $BCDF$ ia a parallelogram, we have $|CD| = |BF|$.

From the figure we see that $|EB| + |BF| = |EF| = 5 + 16 + 9 = 30$.

If we combine these facts, we find that three times the side length of the regular hexagon equals

$$|AB| + |BC| + |CD| = 27 + |EB| + |BF| = 27 + 30 = 57.$$

The side length therefore equals $\frac{57}{3} = 19$.



C-problems

C1. (a) The number 4132 starts with a '4' and is above average because $2 \cdot 3 \geq 4 + 1$ and $2 \cdot 2 \geq 1 + 3$.

(b) Suppose that $a4bc$ is a 4-digit number that is above average, where a, b, and c are the digits '1', '2', and '3' (possibly in a different order). Then $2 \cdot b \geq a + 4 \geq 5$. So $b \geq 3$.
Similarly, we find that $2 \cdot c \geq 4 + b \geq 7$, hence $c \geq 4$. However, this is impossible because c was at most 3.

(c) The numbers 1243756, 1234576, and 1234567 are above average and have digit '7' in the fifth, sixth, and seventh position, respectively.

Digit '7' cannot be in the first position. Indeed, suppose that $7abcdef$ would be above average. Then $2 \cdot b \geq 7 + a \geq 8$, hence $b \geq 4$. Then we must have $2 \cdot c \geq a + b \geq 5$, hence $c \geq 3$. Now we find (in turn) that also $d, e, f \geq 4$. It follows that both digit '1' and digit '2' must be in the position of a, which is impossible.

Digit '7' cannot be in the second or third position. Indeed, otherwise the digit following '7' must be at least 4, which implies that also the digits following it must be at least 4. Digits '1', '2', and '3' must therefore all be in the first two positions, which is impossible.

Finally, digit '7' cannot be in the fourth position. Digit '1' cannot be in the third position since $2 \cdot 1 < 2 + 3$. Because the digit in the third position must be at least 2, the digit in the fifth position must be at least 5. The next digit must therefore be at least 6, as must be the digit following it. The digits '1' to '4' must therefore all be in the first three positions, which is impossible.

C2. (a) Since $x \geq 5$ is odd, we can write $x = 2n + 1$ for an integer $n \geq 2$. Now

$$(x, y, z) = (2n + 1, n, n + 2) \quad \text{and} \quad (x, y, z) = (2n + 1, n + 1, n - 1)$$

are two different good triples. In both cases it is clear that y and z are indeed positive integers and that $y \geq 2$. Substitution into the equation shows that they are indeed solutions:

$$(2n + 1)^2 - 3n^2 = n^2 + 4n + 1 = (n + 2)^2 - 3 \quad \text{and}$$
$$(2n + 1)^2 - 3(n + 1)^2 = n^2 - 2n - 2 = (n - 1)^2 - 3.$$

Thic concludes the solution.

Remark. *One way to arrive at the idea of considering these triples is the following. First substitute $x = 5$. It is then easy to see that z can be no more than 5. The case $z = 5$ is not possible, because then $y = 1$, which is not allowed. Hence z is at most 4. For $z = 1, \ldots, 4$ compute the corresponding value of y, if it exists. This way, you will find two good triples with $x = 5$. Repeating this for $x = 7$ and $x = 9$, you will find good triples as well. The triples found show a clear pattern: when x increases by 2, y and z both increase by 1. This holds for both series of triples. Using this, you can guess a general expression for y and z when $x = 2n + 1$. Checking that the found triples are good by substitution in the equation will then suffice for a complete solution.*

(b) An example is triple $(16, 9, 4)$. This triple is good because $16^2 - 3 \cdot 9^2 = 13 = 4^2 - 3$.

Remark. *Stating a suitable triple and showing that it is a good triple suffices for a complete solution. To find such a triple, one possibility is to take the following approach. Rewrite the equation as $x^2 - z^2 = 3y^3 - 3$. Both sides of the equation can be factored, which gives you : $(x - z)(x + z) = 3(y - 1)(y + 1)$. This will help in finding triples by substituting different values for y. For example, you can try $y = 4$. Then the right-hand side becomes $3 \cdot 3 \cdot 5$, hence the left-hand side becomes $5 \cdot 9$, $3 \cdot 15$, or $1 \cdot 45$. The value of x will always be the average of the two factors, so $x = 7$, $x = 9$, and $x = 23$ in these three cases. There are no even values for x when $y = 4$. If you try further values of y, you will find even values of x for $y = 7$ and $y = 9$.*

Final Round, September 2013

Problems

For these problems not only the answer is important; you also have to describe the way you solved the problem.

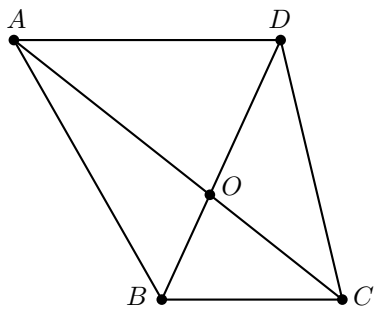
1. In a table consisting of n by n small squares some squares are coloured black and the other squares are coloured white. For each pair of columns and each pair of rows the four squares on the intersections of these rows and columns must not all be of the same colour.
What is the largest possible value of n?

2. Find all triples (x, y, z) of real numbers satisfying

$$x + y - z = -1, \qquad x^2 - y^2 + z^2 = 1 \qquad \text{and} \qquad -x^3 + y^3 + z^3 = -1.$$

3. The sides BC and AD of a quadrilateral $ABCD$ are parallel and the diagonals intersect in O. For this quadrilateral $|CD| = |AO|$ and $|BC| = |OD|$ hold. Furthermore CA is the angular bisector of angle BCD. Determine the size of angle ABC.

Attention: the figure is not drawn to scale. You have to write down your reasoning step by step in text and formulas. No points will be awarded for annotations in a picture alone.



4. For a positive integer n the number $P(n)$ is the product of the positive divisors of n. For example, $P(20) = 8000$, as the positive divisors of 20 are 1, 2, 4, 5, 10 and 20, whose product is $1 \cdot 2 \cdot 4 \cdot 5 \cdot 10 \cdot 20 = 8000$.

(a) Find all positive integers n satisfying $P(n) = 15n$.

(b) Show that there exists no positive integer n such that $P(n) = 15n^2$.

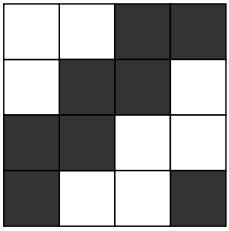
5. The number S is the result of the following sum:

$$1 + 10 + 19 + 28 + 37 + \cdots + 10^{2013}.$$

If one writes down the number S, how often does the digit '5' occur in the result?

Solutions

1. We will prove that $n = 4$ is the largest possible n for which an $n \times n$-table can be coloured according to the rules. The following figure shows a valid colouring for $n = 4$.



Now we prove that there is no colouring of the squares in a 5×5-table satisfying the requirements. Suppose, for contradiction, that such a colouring exists. Of the squares in each row either the majority is black, or the majority is white. We may suppose that there are at least three rows for which the majority of the squares is black (the case where there are at least three rows for which the majority of the squares is white is treated in an analogous way). We now consider the squares in these three rows. Of these 15 squares at least 9 are black.

If there is a column in which each of the three rows has a black square, then each other column can contain at most one black square in these three rows. The total number of black squares in the three rows will therefore be no more than $3+1+1+1+1 = 7$, contradicting the fact that the number should be at least 9.

Hence, in each column at most two of the three rows have a black square. We consider the number of columns with two black squares in the three rows. If there are more then three, then there are two columns in which the same two rows have a black square, which is impossible. It follows that the number of black squares in the three rows is no more than $2+2+2+1+1 = 8$, again contradicting the fact that this number should be at least 9.

Hence, it is impossible to colour a 5×5-table according to the rules. It is clear that it also will be impossible to colour an $n \times n$-table according to the rules if $n > 5$. $\square$

2. The first equation yields $z = x+y+1$. Substitution into the second equation gives $x^2 - y^2 + (x+y+1)^2 = 1$. Expanding gives $2x^2 + 2xy + 2x + 2y = 0$, or $2(x+y)(x+1) = 0$. We deduce that $x+y = 0$ or $x+1 = 0$. We consider the two cases.

- If $x + y = 0$, then $y = -x$ holds. The first equation becomes $z = 1$. Substitution into the third equation yields $-x^3 + (-x)^3 + 1^3 = -1$, or $x^3 = 1$. We deduce that $(x, y, z) = (1, -1, 1)$.
- If $x + 1 = 0$, then $x = -1$ holds. The first equation becomes $z = y$. Substitution into the third equation yields $-(-1)^3 + y^3 + y^3 = -1$, or $y^3 = -1$. Hence, we have $(x, y, z) = (-1, -1, -1)$.

In total we have found two possible solutions. Substitution into the original equations shows that these are indeed both solutions to the system. □

3. First, we prove that some triangles in the figure are isosceles (the top angle coincides with the middle letter).

(1) Triangle ADC is isosceles, because $\angle DAC = \angle ACB = \angle ACD$. The first equality holds because AD and BC are parallel and the second equality follows from the fact that AC is the interior angle bisector of angle BCD.

(2) Triangle DAO is isosceles, because $|AD| = |CD| = |AO|$. The first equality follows from (1) and the second equality is given in the problem statement.

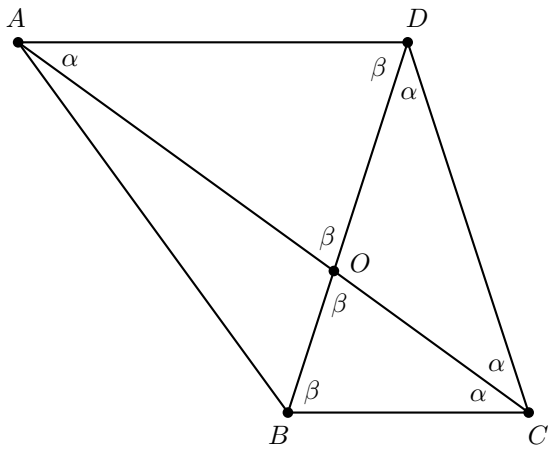
(3) Triangle BCO is isosceles, because it is similar to triangle DAO (hourglass shape).

(4) Triangle COD is isosceles, because $|DO| = |BC| = |CO|$. Here the first equality is given in the problem statement and the second one follows from (3).

(5) Triangle BDC is isosceles, because it is similar to triangle BCO as two pairs of corresponding angles are equal: $\angle DBC = \angle OBC$ and $\angle BDC = \angle DCO = \angle OCB$. Here the last equality follows from (4).

(6) Triangle ADB is isosceles, because $|AD| = |CD| = |BD|$ because of (1) and (5).

Denote $\angle ACB = \alpha$ and $\angle CBD = \beta$ (in degrees). From (5) it follows that $2\alpha = \beta$. From (3) it follows that $180° = 2\beta + \alpha = 5\alpha$, hence $\alpha = \frac{180°}{5} = 36°$ and $\beta = 72°$. In the isosceles triangle ADB the top angle is equal to $\angle ADB = \beta$, hence its equal base angles are $\frac{180°-72°}{2} = 54°$. The requested angle therefore equals $\angle ABC = \angle ABD + \angle DBC = 54° + 72° = 126°$.



□

4. a) Because $P(n) = 15n$ is the product of the positive divisors of n, the prime divisors 3 and 5 of $P(n)$ must also be divisors of n. It follows that n is a multiple of 15. If $n > 15$, then 3, 5, 15 and n are distinct divisors of n, yielding $P(n) \geq 3 \cdot 5 \cdot 15 \cdot n = 225n$. This contradicts the fact that $P(n) = 15n$. The only remaining possibility is $n = 15$. This is indeed a solution, because $P(15) = 1 \cdot 3 \cdot 5 \cdot 15 = 15 \cdot 15$.

b) Suppose that $P(n) = 15n^2$ holds. Again, we find that n is a multiple of 15. It is clear that $n = 15$ is not a sulution, hence $n \geq 30$. We observe that $\frac{n}{5} > 5$. It follows that $1 < 3 < 5 < \frac{n}{5} < \frac{n}{3} < n$ are six distinct divisors of n. Thus $P(n) \geq 1 \cdot 3 \cdot 5 \cdot \frac{n}{5} \cdot \frac{n}{3} \cdot n = n^3$. Because $n > 15$ holds, we have $P(n) \geq n \cdot n^2 > 15n^2$, which contradicts the assumption of this problem. We conclude that no n exists for which $P(n) = 15n^2$. □

5. *To illustrate the idea, we first calculate the number of fives in the result s of the sum $1+10+19+\cdots+100000$. First notice that each term in the sum is a multiple of nine plus 1:*

$$s = 1 + (1+9) + (1+2\cdot 9) + \cdots + (1+11111\cdot 9).$$

The number of terms in the sum is $11111+1 = 11112$ and the average value of a term is $\frac{1+100000}{2}$. It follows that $s = 11112 \cdot \frac{100001}{2} = \frac{11112}{2} \cdot 100001$. Because $\frac{11112}{2} = \frac{5\cdot 11112}{5\cdot 2} = \frac{55560}{10} = 5556$, we find that $s = 5556 + 555600000 = 555605556$. Hence, the number of fives is equal to 6 in this case.

Now we will solve the actual problem. Remark that $10^{2013} = 1 + 9\cdot 11\ldots 1$ (2013 ones). For simplicity let $n = 11\ldots 1$ be the number consisting of 2013 ones. We see that the sum

$$S = 1 + (1+9) + (1+2\cdot 9) + \cdots + (1+n\cdot 9)$$

has exactly $n+1$ terms, with an average value of $\frac{1+10^{1023}}{2}$. Hence $S = \frac{n+1}{2}\cdot(1+10^{2013})$.

Calculating the fraction $\frac{n+1}{2}$ gives:

$$\frac{n+1}{2} = \frac{5n+5}{10} = \frac{555\ldots 560}{10} = 555\ldots 56,$$

a number with 2011 fives followed by a 6. Because the last 2013 digits of the number $10^{2013}\cdot\frac{n+1}{2}$ are all zeroes, there is no 'overlap' between the non-zero digits of $\frac{n+1}{2}$ and $10^{2013}\cdot\frac{n+1}{2}$. We deduce that

$$\begin{aligned} S &= \tfrac{n+1}{2}\cdot(1+10^{2013}) \\ &= \tfrac{n+1}{2} + 10^{2013}\cdot\tfrac{n+1}{2} \\ &= 55\ldots 56055\ldots 56. \end{aligned}$$

Hence S is a number that contains exactly $2011 + 2011 = 4022$ fives. □

BxMO/EGMO Team Selection Test, March 2014

Problems

1. Find all non-negative integers n for which there exist integers a and b such that $n^2 = a + b$ and $n^3 = a^2 + b^2$.

2. Find all functions $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ for which

$$xf(xy) + f(-y) = xf(x)$$

for all non-zero real numbers x, y.

3. In triangle ABC, I is the centre of the incircle. There is a circle tangent to AI at I which passes through B. This circle intersects AB once more in P and intersects BC once more in Q. The line QI intersects AC in R. Prove that $|AR| \cdot |BQ| = |PI|^2$.

4. Let $m \geq 3$ and n be positive integers such that $n > m(m-2)$. Find the largest positive integer d such that $d \mid n!$ and $k \nmid d$ for all $k \in \{m, m+1, \ldots, n\}$.

5. Let n be a positive integer. Daniël and Merlijn are playing a game. Daniël has k sheets of paper lying next to each other on a table, where k is a positive integer. On each of the sheets, he writes some of the numbers from 1 up to n (he is allowed to write no number at all, or all numbers). On the back of each of the sheets, he writes down the remaining numbers. Once Daniël is finished, Merlijn can flip some of the sheets of paper (he is allowed to flip no sheet at all, or all sheets). If Merlijn succeeds in making all of the numbers from 1 up to n visible at least once, then he wins.
Determine the smallest k for which Merlijn can always win, regardless of Daniël's actions.

Solutions

1. By AM-GM applied to a^2 and b^2, we find that $a^2 + b^2 \geq 2ab$. As $2ab = (a+b)^2 - (a^2+b^2)$, it follows that $n^3 \geq (n^2)^2 - n^3$, i.e. $2n^3 \geq n^4$. Hence either $n = 0$ or $2 \geq n$, so $n = 0$, $n = 1$, and $n = 2$ are the only possibilities. For $n = 0$ we find $a = b = 0$ as solution, for $n = 1$ we find $a = 0$, $b = 1$ as solution, and for $n = 2$ we find $a = b = 2$ as solution. Thus the non-negative integers n with the desired property are precisely $n = 0$, $n = 1$, and $n = 2$.□

2. Substituting $x = 1$ gives $f(y) + f(-y) = f(1)$ for all y, or equivalently, $f(-y) = f(1) - f(y)$ for all y. Substituting $y = -1$ then gives $xf(-x) + f(1) = xf(x)$ for all x. We now substitute $f(-x) = f(1) - f(x)$ to obtain $x\big(f(1) - f(x)\big) + f(1) = xf(x)$, so $xf(1) + f(1) = 2xf(x)$. We see that f is of the form $f(x) = c + \frac{c}{x}$ for a certain $c \in \mathbb{R}$.

We check whether this family of functions satisfies the equation. The left hand side now reads $xf(xy) + f(-y) = x(c + \frac{c}{xy}) + c + \frac{c}{-y} = xc + c$, and the right hand side reads $xf(x) = x(c + \frac{c}{x}) = xc + c$. Hence this function satisfies the given equation for all $c \in \mathbb{R}$. □

3. There is only one configuration. We have

$$\begin{aligned} \angle AIP &= \angle IBP && \text{(inscribed angle theorem)} \\ &= \angle IBQ && (IB \text{ is angle bisector}) \\ &= \angle IPQ && \text{(quadrilateral } PBQI \text{ is cyclic)} \end{aligned}$$

hence $AI \parallel PQ$. This implies $\angle IAB = \angle QPB = \angle QIB$, using the cyclic quadrilateral for the latter equality. We have already seen that $\angle AIP = \angle IBQ$, so $\triangle IAP \sim \triangle BIQ$ (aa). Hence

$$\frac{|AP|}{|PI|} = \frac{|QI|}{|BQ|}. \tag{1}$$

Moreover, $\angle RIA$ is the angle opposite to an inscribed angle, hence equal to $\angle IPQ$, which we already know to be equal to $\angle AIP$. Hence $\angle RIA = \angle AIP$. As AI is an angle bisector, we have $\angle RAI = \angle PAI$, so $\triangle RAI \cong \triangle PAI$ (ASA). Hence $|AR| = |AP|$. Moreover, I is the centre of the arc PQ as BI is an angle bisector, so $|PI| = |QI|$. Now (1) gives

$$\frac{|AR|}{|PI|} = \frac{|PI|}{|BQ|},$$

implying that $|AR| \cdot |BQ| = |PI|^2$, as desired. □

4. We prove that $d = m - 1$ is the largest integer satisfying the conditions. First note that $m-1 \mid n!$ and that for $k \geq m$ we have $k \nmid m-1$, so $d = m-1$ indeed satisfies the conditions.

Now suppose that for some d we have $d \mid n!$ and $k \nmid d$ for all $k \in \{m, m+1, \ldots, n\}$. We prove that $d \leq m - 1$. Write $d = p_1 p_2 \cdots p_t$, where the p_i are prime for all i (but not necessarily pairwise distinct). If $t = 0$, then $d = 1 \leq m - 1$, so we may assume that $t \geq 1$. From the first condition on d, it follows that $p_i \leq n$ for all i. From the second condition on d, it follows that $p_i \notin \{m, m+1, \ldots, n\}$ for all i. Hence $p_i \leq m-1$ for all i. Now consider the integers $p_1, p_1 p_2, \ldots, p_1 p_2 \cdots p_t$. These are divisors of d and hence all are not in $\{m, m+1, \ldots, n\}$. Moreover, we know that $p_1 \leq m-1$. Consider the largest $j \leq t$ such that $p_1 p_2 \cdots p_j \leq m - 1$. If $j < t$, then

$$p_1 p_2 \cdots p_j p_{j+1} \leq (m-1) p_{j+1} \leq (m-1)(m-1) = m(m-2) + 1 \leq n.$$

But then $p_1 p_2 \cdots p_j p_{j+1} \leq m - 1$, contradicting the maximality of j. Hence $j = t$, so $d = p_1 p_2 \cdots p_t \leq m - 1$.

We conclude that $d = m - 1$ is indeed the largest integer satisfying the conditions. □

5. We give each of Daniël's sheets of paper a different colour. Moreover, we have n boxes with the numbers from 1 up to n on them. We make sure we have enough chips in the colours of Daniël's sheets. For each sheet, Merlijn puts a chip with the colour of this sheet in every box of which the number is on the front of this sheet. So every box will contain precisely those chips of which the colours are those of the sheets on which this number is on the front.

For each sheet of paper Merlijn flips, he takes a chip of the same colour from the supply. A number is not visible on the table if and only if the sheets on which this number was on the front, are precisely the sheets that Merlijn flipped, i.e. if and only if the set of chips that he took is equal to the set of chips in the box with this number. Hence Merlijn wins if and only if the set of chips he took does not occur in any of the boxes, since then all of the numbers are visible. Now we claim that Merlijn can win if and only if $2^k > n$. So the smallest k for which Merlijn can win, is the smallest k such that $2^k > n$.

Suppose $2^k > n$. The number of possible sets of colours is 2^k, hence larger than n. There are n boxes, so not every set of colours can occur in the boxes. Hence Merlijn can pick a set of colours not occurring in any of the boxes, flip the corresponding sheets, and win.

Now suppose that $2^k \leq n$. Then Daniël first fills the boxes with sets of chips, in such a way that every possible set of colours occurs in some box. This is possible, since there are 2^k possible sets of colours, and we have at least 2^k boxes. Now Daniël writes on the front of each sheet of coloured paper precisely those numbers whose boxes contain the colour of that sheet, and on the back the remainder of the numbers. In this way, the chips in the boxes correspond to the numbers on the fronts of the sheets. Now Merlijn cannot choose a set of colours not occurring in any of the boxes; he cannot win. $\square$

IMO Team Selection Test 1, June 2014

Problems

1. Determine all pairs (a, b) of positive integers satisfying

$$a^2 + b \mid a^2 b + a \qquad \text{and} \qquad b^2 - a \mid ab^2 + b.$$

2. Let $\triangle ABC$ be a triangle. Let M be the midpoint of BC and let D be a point on the interior of side AB. The intersection of AM and CD is called E. Suppose that $|AD| = |DE|$. Prove that $|AB| = |CE|$.

3. Let a, b and c be rational numbers for which $a + bc$, $b + ac$ and $a + b$ are all non-zero and for which we have

$$\frac{1}{a + bc} + \frac{1}{b + ac} = \frac{1}{a + b}.$$

Prove that $\sqrt{(c-3)(c+1)}$ is rational.

4. Let $\triangle ABC$ be a triangle with $|AC| = 2|AB|$ and let O be its circumcentre. Let D be the intersection of the angle bisector of $\angle A$ and BC. Let E be the orthogonal projection of O on AD and let $F \neq D$ be a point on AD satisfying $|CD| = |CF|$. Prove that $\angle EBF = \angle ECF$.

5. On each of the 2014^2 squares of a 2014×2014-board a light bulb is put. Light bulbs can be either on or off. In the starting situation a number of the light bulbs is on. A move consists of choosing a row or column in which at least 1007 light bulbs are on and changing the state of all 2014 light bulbs in this row or column (from on to off or from off to on). Find the smallest non-negative integer k such that from each starting situation there is a finite sequence of moves to a situation in which at most k light bulbs are on.

Solutions

1. From $a^2 + b \mid a^2b + a$ it follows that

$$a^2 + b \mid (a^2b + a) - b(a^2 + b) = a - b^2.$$

From $b^2 - a \mid ab^2 + b$ it follows that

$$b^2 - a \mid (ab^2 + b) - a(b^2 - a) = b + a^2.$$

Hence we have $a^2 + b \mid a - b^2 \mid a^2 + b$. This means that $a^2 + b$ is equal to $a - b^2$, up to sign. We distinguish two cases: $a^2 + b = b^2 - a$ and $a^2 + b = a - b^2$. In the latter case we have $a^2 + b^2 = a - b$. But $a^2 \geq a$ and $b^2 \geq b > -b$, hence this is impossible. Therefore we must be in the former case: $a^2 + b = b^2 - a$. This yields $a^2 - b^2 = -a - b$, hence $(a + b)(a - b) = -(a + b)$. As $a + b$ is positive, we may divide by it and we get $a - b = -1$, hence $b = a + 1$. All pairs that could possibly satisfy the conditions are of the form $(a, a + 1)$ for a positive integer a.

We consider these pairs. We have $a^2 + b = a^2 + a + 1$ and $a^2b + a = a^2(a+1) + a = a^3 + a^2 + a = a(a^2 + a + 1)$, hence the first divisibility condition is satisfied. Furthermore, we have $b^2 - a = (a + 1)^2 - a = a^2 + a + 1$ and $ab^2 + b = a(a+1)^2 + (a+1) = a^3 + 2a^2 + 2a + 1 = a(a^2 + a + 1) + a^2 + a + 1 = (a+1)(a^2 + a + 1)$, hence also the second divisibility condition is satisfied. Hence the pairs $(a, a + 1)$ satisfy the conditions and they are exactly the pairs satisfying the conditions. □

2. We apply Menelaos's theorem to the line through A, E and M inside triangle BCD. This yields

$$\frac{|BM|}{|MC|} \cdot \frac{|CE|}{|ED|} \cdot r\frac{|DA|}{|AB|} = 1.$$

Because M is the midpoint of BC, we have $\frac{|BM|}{|MC|} = 1$. Furthermore, it is given that $|AD| = |DE|$. Altogether this yields $|CE| = |AB|$. □

3. We have

$$\frac{1}{a+bc}+\frac{1}{b+ac}=\frac{(b+ac)+(a+bc)}{(a+bc)(b+ac)}=\frac{a+b+ac+bc}{ab+a^2c+b^2c+abc^2}.$$

Hence, the problem statement's equality yields

$$(a+b)(a+b+ac+bc)=ab+a^2c+b^2c+abc^2,$$

or equivalently,

$$a^2+ab+a^2c+abc+ab+b^2+abc+b^2c=ab+a^2c+b^2c+abc^2,$$

or equivalently,

$$a^2+2abc+ab-abc^2+b^2=0.$$

We can consider this as a quadratic equation in a, of which we know that it has a rational solution. Hence, the discriminant of this equation must be the square of a rational number. This discriminant equals

$$\begin{aligned}D&=(2bc+b-bc^2)^2-4b^2\\&=(2bc+b-bc^2-2b)(2bc+b-bc^2+2b)\\&=b^2(2c-1-c^2)(2c+3-c^2)\\&=b^2(c^2-2c+1)(c^2-2c-3)\\&=b^2(c-1)^2(c+1)(c-3).\end{aligned}$$

If $c=1$, then the original equation becomes

$$\frac{1}{a+b}+\frac{1}{a+b}=\frac{1}{a+b},$$

which has no solution. If $a=0$, then the original equation becomes

$$\frac{1}{bc}+\frac{1}{b}=\frac{1}{b},$$

which also has no solution. Hence, $c\neq 1$ and $a\neq 0$. In particular,

$$(c-3)(c+1)=\frac{D}{b^2(c-1)^2}$$

must be the square of a rational number. □

4. Let $G\neq A$ be the intersection of AD with the circumcircle of $\triangle ABC$. Then we have $|AG|=2|AE|$ because the projection of the centre of a circle

on a chord of this circle is the midpoint of that chord. Let M be the midpoint of AC. Because $|AB| = \frac{|AC|}{2} = |AM|$ and because AD is the angle bisector of $\angle BAM$, we have that M is the image of B under reflection in AD. Now we have $\angle DGC = \angle AGC = \angle ABC = \angle ABD = \angle DMA = 180° - \angle DMC$, hence $DMCG$ is a cyclic quadrilateral. Then we have $AM^2 = \frac{AM \cdot AC}{2} = \frac{AD \cdot AG}{2} = AD \cdot AE$, hence $\triangle AME \sim \triangle ADM$ (SAS). Now we have $180° - \angle EMC = \angle EMA = \angle MDA = \angle BDA = \angle CDF = \angle DFC = \angle EFC$, hence $EMCF$ is a cyclic quadrilateral. Hence, $\angle EBF = \angle EMF = \angle ECF$, which is exactly what we needed to prove. □

5. Number the rows from 1 up to 2014 and also number the columns. Consider the following beginning situation: in row i the light bulbs in columns i, $i+1, \ldots, i+1005$ are on and the rest are off, in which we take the column numbers modulo 2014. In each row and in each column there are exactly 1006 light bulbs that are on. Hence, there is no move possible. It is not always possible to get to a situation in which less than $2014 \cdot 1006$ light bulbs are on.

Now we shall show that it is always possible to get to a situation in which at most $2014 \cdot 1006$ are on. Suppose, from the contrary, that in a certain situation at least $2014 \cdot 1006 + 1$ light bulbs are on and it is not possible to get to a situation in which less bulbs are on. If there is a row or column in which at least 1008 bulbs are on, then we can change the states of these bulbs in this row or column and there will be less bulbs that are on. This is a contradiction, hence in each row and column at most 1007 bulbs are on.

Let I be the set of rows in which exactly 1007 bulbs are on and let J_1 be the set of columns in which exactly 1007 bulbs are in. We will try to change the state of the bulbs in all rows in I. This we call *the big plan.* If after executing the big plan there is a column in which at least 1008 bulbs are on, then we get a contradiction. Hence we shall assume that this does not happen. Let J_2 be the set of columns that, after executing the big plan, contain exactly 1007 bulbs that are on. If there exists a square (i, j) with $i \in I$ and $j \in J_1$ containing a bulb that is off, we can switch row i and then column j gets more than 1007 bulbs that are on, which is a contradiction. Hence every bulb on (i, j) with $i \in I$ and $j \in J_1$ is on. If there exists a square (i, j) with $i \in I$ and $j \in J_2$ containing a bulb that is off, after executing the big plan, then we get a contradiction in the same way. Hence every bulb on (i, j) with $i \in I$ and $j \in J_2$ is off after executing the big plan. Because the columns in J_2 contain exactly 1007 bulbs that are on after executing the big plan, there are $1007 - |I|$ bulbs that are on before executing the big plan. *(For a set X, we denote by $|X|$ its number*

of elements.) In the columns of J_1 there are exactly 1007 that are on, and this also means that J_1 and J_2 are disjoint. In the other columns at most 1006 bulbs are on. The total number of bulbs that are on before the big plan is at most

$$(1007-|I|)|J_2|+1007|J_1|+1006(2014-|J_1|-|J_2|) = 1006\cdot 2014+|J_1|+|J_2|-|I|\cdot|J_2|.$$

This has to be at least $1006 \cdot 2014 + 1$, hence $|J_1| + |J_2| - |I| \cdot |J_2| \geq 1$. This yields $|J_1| > (|I| - 1)|J_2|$. If $|I| \geq 2$, then $|J_1| > |J_2|$. By executing the big plan, the number of columns in which 1007 bulbs are on decreases. But after that, we again have a situation with $|I|$ rows containing 1007 bulbs that are on, and in which J_1 and J_2 have been interchanged. Then we can again apply the big plan, to decrease the number of columns that contain 1007 bulbs that are on again. This is a contradiction, because now we are back in the beginning situation. We conclude that we must have $|I| = 1$.

We have a situation in which there is exactly one row with exactly 1007 bulbs that are on. Analogously, we can show that there must be exactly one column in which exactly 1007 bulbs are on. Because there are $1006\cdot 2014+1$ bulbs that are on, each other row and column must contain exactly 1006 bulbs that are on. Now change the state of the bulbs in the row with 1007 bulbs that are on. In 1007 columns there will be $1006 + 1 = 1007$ bulbs that are on. We have already seen that in this situation we can decrease the number of bulbs that are on.

We conclude that if there are more than $1006 \cdot 2014$ bulbs that are on, it is always possible to decrease this number. Hence, the smallest k is $1006 \cdot 2014$. □

IMO Team Selection Test 2, June 2014

Problems

1. Let $f\colon \mathbb{Z}_{>0} \to \mathbb{R}$ be a function such that for all $n > 1$ there is a prime divisor p of n such that

$$f(n) = f\left(\frac{n}{p}\right) - f(p).$$

Furthermore, it is given that $f(2^{2014}) + f(3^{2015}) + f(5^{2016}) = 2013$. Determine $f(2014^2) + f(2015^3) + f(2016^5)$.

2. The sets A and B are subsets of the positive integers. The sum of any two distinct elements of A is an element of B. The quotient of any two distinct elements of B (where we divide the largest by the smallest of the two) is an element of A. Determine the maximum number of elements in $A \cup B$.

3. Let H be the orthocentre of an acute triangle ABC. The line through A perpendicular to AC and the line through B perpendicular to BC intersect in D. The circle with centre C through H intersects the circumcircle of triangle ABC in the points E and F. Prove that $|DE| = |DF| = |AB|$.

4. Determine all pairs (p, q) of prime numbers for which $p^{q+1} + q^{p+1}$ is a square.

5. Let $P(x)$ be a polynomial of degree $n \leq 10$ with integral coefficients such that for every $k \in \{1, 2, \ldots, 10\}$ there is an integer m with $P(m) = k$. Furthermore, it is given that $|P(10) - P(0)| < 1000$. Prove that for every integer k there is an integer m such that $P(m) = k$.

Solutions

1. If $n = q$ with q prime, then there is only one prime divisor of n, namely q, hence we must have that $f(q) = f(1) - f(q)$, hence $f(q) = \frac{1}{2}f(1)$. If $n = q^2$ with q prime, then n also has only one prime divisor, hence we have $f(q^2) = f(q) - f(q) = 0$. We will prove by induction to k that $f(q^k) = \frac{2-k}{2}f(1)$ if q is a prime number and k a positive integer. For $k = 1$ and $k = 2$ we have already shown this. Now suppose that $f(q^k) = \frac{2-k}{2}f(1)$ for certain $k \geq 2$ and substitute $n = q^{k+1}$. Then we have

$$f(q^{k+1}) = f(q^k) - f(q) = \frac{2-k}{2}f(1) - \frac{1}{2}f(1) = \frac{2-(k+1)}{2}f(1).$$

This completes the induction argument.

Now we will use the second equality. We have

$$\begin{aligned} 2013 &= f(2^{2014}) + f(3^{2015}) + f(5^{2016}) \\ &= \frac{2-2014}{2}f(1) + \frac{2-2015}{2}f(1) + \frac{2-2016}{2}f(1) \\ &= -\frac{6039}{2}f(1), \end{aligned}$$

hence $f(1) = \frac{2013 \cdot 2}{-6039} = -\frac{2}{3}$. Then we have for each prime number q that $f(q) = \frac{1}{2}f(1) = -\frac{1}{3}$.

We will prove the following statement: if $n = p_1 p_2 \cdots p_m$ with $p_1, p_2, \ldots, p_m$ not necessarily distinct prime numbers and $m \geq 0$, then we have $f(n) = \frac{m-2}{3}$. This we will do by induction to m. For $m = 0$ we have $n = 1$ and $f(1) = -\frac{2}{3} = \frac{0-2}{3}$, hence in this case it is true. Now suppose that we have proved the induction hypothesis for certain $m \geq 0$. Consider an arbitrary n of the form $n = p_1 p_2 \cdots p_{m+1}$. Then $n > 1$, hence there is a prime factor $p \mid n$ for which we have $f(n) = f(\frac{n}{p}) - f(p)$; without loss of generality this is $p = p_{m+1}$. Now it follows that

$$f(n) = f(p_1 p_2 \cdots p_m) - f(p_{m+1}) = \frac{m-2}{3} - -\frac{1}{3} = \frac{(m+1)-2}{3}.$$

This completes the induction.

Now we can calculate the answer. The prime factorisations of 2014, 2015 and 2016 are $2014 = 2 \cdot 19 \cdot 53$, $2015 = 5 \cdot 13 \cdot 31$ and $2016 = 2^5 \cdot 3^2 \cdot 7$, hence

$$f(2014^2) + f(2015^3) + f(2016^5) = \frac{6-2}{3} + \frac{9-2}{3} + \frac{40-2}{3} = \frac{49}{3}.$$

□

2. Suppose that A contains at least three elements, say $a < b < c$. Then B contains the three distinct elements $a+b < a+c < b+c$. Hence, A certainly contains the element $\frac{b+c}{a+c}$. Apparently this fraction is an integer, hence $a+c \mid b+c$. But then it follows that $a+c \mid (b+c)-(a+c) = b-a$. We know that $b > a$, hence $b-a$ is positive, hence we must have $a+c \leq b-a$. This yields $c \leq b-2a < b$, which is in contradiction with $c > b$. Hence, A contains at most two elements.

Suppose that B contains at least four elements, say $a < b < c < d$. Then A contains the three distinct elements $\frac{d}{a}$, $\frac{d}{b}$ and $\frac{d}{c}$. But A cannot contain three distinct elements, contradiction. Hence, B contains at most three elements.

In total $A \cup B$ contains at most 5 elements. This number can be attained, take for example $A = \{2,4\}$ and $B = \{3,6,12\}$. Now $2+4 = 6 \in B$ and $\frac{12}{6} = \frac{6}{3} = 2 \in A$ and $\frac{12}{3} = 4 \in A$, hence this pair of sets satisfies the conditions. We conclude that the maximum number of elements of $A \cup B$ is 5. □

3. The triangle is acute, hence H lies inside the triangle. This means that E and F lie on the short arcs AC and BC. Suppose that E lies on the short arc AC and that F lies on the short arc BC.

If we reflect H in AC, then the reflection H' lies on the circumcircle of $\triangle ABC$. (This is a well-known fact, which can be proved by angle chasing to prove that $\angle AHC = 180° - \angle ABC$.) On the other hand, this reflection also lies on the circle with centre C through H, because $|CH'| = |CH|$. Hence, H' is the intersection point of the two circles and that is E. We conclude that E is the image of H under the reflection in AC.

This means that EH is perpendicular to AC and hence it is the same line as BH. Because AD is also perpendicular to AC, the lines BE are AD parallel. Furthermore, D lies on the circumcircle of $\triangle ABC$ because $\angle CAD + \angle CBD = 90° + 90° = 180°$. We have already seen that E lies on the short arc AC, hence $EADB$ is a cyclic quadrilateral (in this order). Now we have $\angle BEA + \angle EAD = 180°$ since BE and AD are parallel, but also $\angle EBD + \angle EAD = 180°$ because of the cyclic quadrilateral. Hence, $\angle BEA = \angle EBD$, hence the corresponding chords BA and ED have the same length.

Analogously we can prove that $|AB| = |DF|$, which solves the problem. □

4. First suppose that both p and q are odd. Then the exponents in the sum $p^{q+1} + q^{p+1}$ are both even, from which it follows that both terms are congruent to 1 mod 4. Hence, the sum is congruent to 2 mod 4, but this is never a square.

Now suppose that both p and q are even. Then, they are both equal to 2. That yields $p^{q+1} + q^{p+1} = 2^3 + 2^3 = 16 = 4^2$, hence this pair satisfies the conditions.

Finally, suppose that one of both, say p, is even and the other one is odd. We then have $p = 2$ and $2^{q+1} + q^3 = a^2$ for a certain positive integer a. Write $q + 1 = 2b$ with b a positive integer, then the equality becomes $2^{2b} + q^3 = a^2$, or equivalently

$$q^3 = a^2 - 2^{2b} = (a - 2^b)(a + 2^b).$$

Both factors on the right hand side now have to be a power of q, say $a - 2^b = q^k$ and $a + 2^b = q^l$ with $l > k \geq 0$. Both factors are divisible by q^k, hence also the difference is divisible by it. Hence, $q^k \mid 2 \cdot 2^b = 2^{b+1}$. However, q is an odd prime, hence the only power of q that is a divisor of a power of two, is 1. Hence, $k = 0$. Now we get $q^3 = a + 2^b$ and $a - 2^b = 1$, hence $q^3 = (2^b + 1) + 2^b = 2^{b+1} + 1$. This yields

$$2^{b+1} = q^3 - 1 = (q - 1)(q^2 + q + 1).$$

However, $q^2 + q + 1 \equiv 1 \mod 2$ and furthermore $q^2 + q + 1 > 1$, hence this can never be a power of two. Contradiction.

We conclude that $(p, q) = (2, 2)$ is the only solution. $\square$

5. For $i = 1, 2, \ldots, 10$ let c_i be an integer such that $P(c_i) = i$. For $i \in \{1, 2, \ldots, 9\}$ we have that

$$c_{i+1} - c_i \mid P(c_{i+1}) - P(c_i) = (i+1) - i = 1,$$

hence $c_{i+1} - c_i = \pm 1$ for all $i \in \{1, 2, \ldots, 9\}$. Furthermore, it holds that $c_i \neq c_j$ for $i \neq j$, because $P(c_i) = i \neq j = P(c_j)$. We conclude that $c_1, c_2, \ldots, c_{10}$ are ten consecutive integers, either in ascending or descending order. Hence, we will consider the following two cases:

(A) $c_i = c_1 - 1 + i$ for $i = 1, 2, \ldots, 10$ (i.e., $c_1, c_2, \ldots, c_{10}$ is an ascending sequence of consecutive integers),

(B) $c_i = c_1 + 1 - i$ for $i = 1, 2, \ldots, 10$ (i.e., $c_1, c_2, \ldots, c_{10}$ is a descending sequence of consecutive integers).

First consider case (A). Define $Q(x) = 1 + x - c_1$. Then for $1 \leq i \leq 10$ we have that

$$Q(c_i) = Q(c_1 - 1 + i) = 1 + (c_1 - 1 + i) - c_1 = i = P(c_i),$$

hence $P(c_i) - Q(c_i) = 0$. Hence, we can also write

$$P(x) - Q(x) = R(x) \cdot \prod_{i=1}^{10} (x - c_i),$$

or equivalently,

$$P(x) = 1 + x - c_1 + R(x) \cdot \prod_{i=1}^{10} (x - c_i).$$

Because the degree of P is at most 10, the degree of R cannot be greater than 0. Hence $R(x)$ is a constant, say $R(x) = a$ with $a \in \mathbb{Z}$. We then get that

$$P(x) = 1 + x - c_1 + a \cdot \prod_{i=1}^{10} (x - c_i).$$

Now we substitute $x = 10$ and $x = 0$:

$$P(10) - P(0) = 1 + 10 - c_1 + a \cdot \prod_{i=1}^{10} (10 - c_i) - (1 + 0 - c_1) - a \cdot \prod_{i=1}^{10} (0 - c_i)$$

$$= 10 + a \cdot \left(\prod_{i=1}^{10} (10 - c_i) - \prod_{i=1}^{10} (0 - c_i) \right).$$

The numbers $10-c_1$, $10-c_2$, ..., $10-c_{10}$ are ten consecutive numbers and the numbers $0-c_1$, $0-c_2$, ..., $0-c_{10}$ are the next ten consecutive numbers. Hence, there is an N such that

$$\prod_{i=1}^{10}(10-c_i)-\prod_{i=1}^{10}(0-c_i) = (N+20)(N+19)\cdots(N+11)-(N+10)(N+9)\cdots(N+1).$$

We will find a bound for this quantity. First we suppose that $N+1>0$. Then we have

$$\begin{aligned}(N+20)&(N+19)\cdots(N+11)-(N+10)(N+9)\cdots(N+1)\\ &> (N+20)(N+9)\cdots(N+1)-(N+10)(N+9)\cdots(N+1)\\ &= 10\cdot(N+9)(N+8)\cdots(N+1)\\ &\geq 10!.\end{aligned}$$

If $N+20<0$, then all factors are negative. Completely analogously the absolute difference is again much greater than 10!. If $N+20\geq 0$ and $N+1\leq 0$, then one of the factors is 0. Hence, exactly one of the two terms is equal to zero and the other one is at least 10! in absolute value. We conclude that the absolute difference always is at least 10!. Hence, if $a\neq 0$, then $|P(10)-P(0)|\geq 10!-10>1000$. Given is, however, that $|P(10)-P(0)|<1000$. Apparently we must have that $a=0$. Now we find that

$$P(x) = 1+x-c_1.$$

Let $k\in\mathbb{Z}$ be arbitrary and pick $m=k-1+c_1$. Then it holds that $P(m) = 1+(k-1+c_1)-c_1 = k$. Hence for any integer k there is an integer m with $P(m)=k$.

Now consider case (B). We can use the exact same reasoning, in which we now define $Q(x)=1-x+c_1$ and eventually get that

$$P(x) = 1-x+c_1+a\cdot\prod_{i=1}^{10}(x-c_i).$$

In the same way, we deduce that $a=0$, yielding

$$P(x) = 1-x+c_1.$$

And now it follows again that for any integer k there is an integer m with $P(m)=k$. □

Junior Mathematical Olympiad, October 2013

Problems

Part 1

1. The four symbols ◯, ◁, ☆, and □ represent distinct digits. Suppose that

$$\bigcirc \times \bigcirc = \triangleleft\bigcirc \qquad \text{and} \qquad \star + \star = \square\bigcirc.$$

Which digit is ☆?

A) 5 B) 6 C) 7 D) 8 E) 9

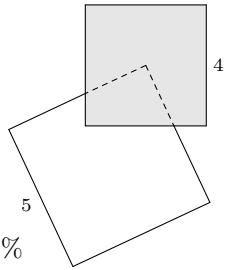
2. The Free family is driving on the German highway to a faraway resort. Their fuel tank is full at the beginning of their journey. A third of the way through the journey, 75% of the fuel is remaining in the fuel tank. How much fuel is remaining in the fuel tank halfway through the journey?

A) 25% B) $33\frac{1}{3}$% C) 50% D) 60% E) 62,5%

3. A football tournament with five teams is held, in which every pair of teams plays one match against each other. Two points are awarded for winning a match, one point is awarded for a draw, and zero points are awarded for losing a match. After the tournament, every team has a number of points different from all of the other teams. What are the possibilities for the number of points of the winning team?

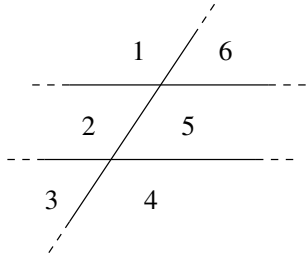
A) 8 B) 8, 7 C) 8, 6 D) 8, 7, 6 E) 8, 7, 6, 5

4. A small 4 by 4 square lies partially on a larger 5 by 5 square, in such a way that one of the vertices of the larger square lies directly beneath the centre of the small square. What part of the large square is covered by the small square?



A) 15% B) 16% C) 17,5%
D) 18% E) This cannot be determined.

5. A large sheet of paper can be divided into at most 6 pieces with two horizontal lines and one other line, see figure. With five horizontal lines and five other lines, what is the maximal number of pieces that you can obtain?



A) 43 B) 44 C) 45 D) 46 E) 47

6. Five children, Ahmed, Bob, Celine, Dan and Eve, are standing in a queue to buy ice cream at an ice cream van. They all have an integer number of euros. The one or more children in front of Ahmed in the queue, together have 4 euros. The children between Celine and Dan together have 7 euros. The children in front of Eve together have 6 euros, and Eve has 2 euros herself. How many euros does Ahmed have?

A) 1 B) 2 C) 3 D) 4 E) 5

7. We split the numbers from 1 up to 9 into a group of four and a group of five. In both groups, we multiply the numbers. Then we divide the larger of the two results by the smaller one. We require the result to be an integer. In how many ways can we split the numbers 1 up to 9 into a group of four and a group of five so that this is the case?

A) 1 B) 2 C) 3 D) 4 E) 5

8. John has six squares of equal size: two red ones, two grey ones, and two blue ones. He makes a cube out of them by gluing them together. How many different cubes can John make? Two cubes are different if they cannot be transformed into one another by a rotation of the cube.

A) 3 B) 4 C) 5 D) 6 E) 8

Part 2

1. What is the smallest number that can be obtained by adding *three* consecutive positive even integers, but also by adding *four* consecutive positive even integers? (A number is *positive* if it is larger than 0.)

2. Nick has red and blue marbles. All red marbles have the same weight, and so do all blue marbles. Nick weighs a red marble and a blue one: together they weigh 30 grams. Then he weighs a number of red marbles and a blue one: they weigh 180 grams. Exactly the same total number of marbles, but now with one red instead of one blue marble, together weigh 60 grams. What is the weight of a red marble in grams?

3. You have square tiles with edges of length 1 metre, 2 metres, 3 metres, and so on. You have more than enough tiles of each kind. What is the length of the side of the smallest square you can make using precisely 11 of these tiles?

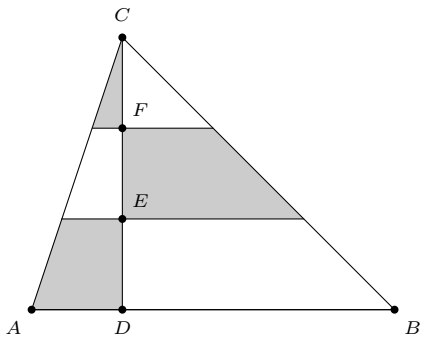
4. How many numbers between 1 and 1000 are not divisible by 2, divisible by 3, not divisible by 5, and divisible by 7?

5. A collection of distinct positive integers satisfies the following. Every triple of integers in this collection adds up to less than 37. Every quadruple of integers in this collection adds up to more than 37. What is the maximum number of integers this collection can contain?

6. In this problem, a, b, c, and d represent digits not equal to 0, in such a way that the calculation to the right is correct. Determine all possibilities for $cdab$.

$$\begin{array}{cccc} a & b & 2 & 0 \\ 1 & 3 & c & d \\ \hline c & d & a & b \end{array} \; -$$

7. In the figure below there is a triangle ABC. The segment AB is horizontal, and the segment CD is vertical. The segment DB is thrice as long as AD. Moreover, E and F divide CD into three parts of equal length. The two horizontal lines through E and F divide, together with CD, the triangle ABC into six parts. Triangle ABC has area 1. What is the combined area of the three grey parts?



8. Ionica and Jeanine cross out four numbers each in the figure to the right, so that exactly one number remains. They each add up the numbers they crossed out. Ionica's sum is thrice as large as Jeanine's sum. Which numbers could be the remaining one?

4	20	18
7	11	9
25	2	6

Solutions

Part 1

1. D) 8

2. E) $62,5\%$

3. D) 8, 7, 6

4. B) 16%

5. D) 46

6. B) 2

7. D) 4

8. D) 6

Part 2

1. 36

2. 25 grams

3. 5 metres

4. 19

5. 6

6. 5367

7. $\frac{5}{12}$

8. 6 and 18

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